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Abstract. Consider the Newton equation in the relativistic case (that is the Newton-Einstein equation)

$$\dot{p} = F(x),\ F(x) = -\nabla V(x),$$
$$p = \frac{\dot{x}}{\sqrt{1-\frac{|\dot{x}|^2}{c^2}}},\ \dot{p} = \frac{dp}{dt},\ \dot{x} = \frac{dx}{dt},\ x \in C^1(\mathbb{R}, \mathbb{R}^d), \tag{$*$}$$

$$\text{where } V \in C^2(\mathbb{R}^d, \mathbb{R}),\ |\partial_x^j V(x)| \le \beta_{|j|}(1+|x|)^{-(\alpha+|j|)}$$

for $|j| \le 2$ and some $\alpha > 1$. We give estimates and asymptotics for scattering solutions and scattering data for the equation $(*)$ for the case of small angle scattering. We show that at high energies the velocity valued component of the scattering operator uniquely determines the X-ray transform PF. Applying results on inversion of the X-ray transform P we obtain that for $d \ge 2$ the velocity valued component of the scattering operator at high energies uniquely determines F. In addition we show that our high energy asymptotics found for the configuration valued component of the scattering operator doesn't determine uniquely F. The results of the present work were obtained in the process of generalizing some results of Novikov [No] to the relativistic case.

1. Introduction

Consider the Newton equation in the relativistic case (that is the Newton-Einstein equation)

$$\dot{p} = F(x),\ F(x) = -\nabla V(x),$$
$$p = \frac{\dot{x}}{\sqrt{1-\frac{|\dot{x}|^2}{c^2}}},\ \dot{p} = \frac{dp}{dt},\ \dot{x} = \frac{dx}{dt},\ x \in C^1(\mathbb{R}, \mathbb{R}^d), \tag{1.1}$$

$$\text{where } V \in C^2(\mathbb{R}^d, \mathbb{R}),\ |\partial_x^j V(x)| \le \beta_{|j|}(1+|x|)^{-(\alpha+|j|)} \tag{1.2}$$

for $|j| \le 2$ and some $\alpha > 1$ (here j is the multiindex $j \in (\mathbb{N} \cup \{0\})^d, |j| = \sum_{n=1}^d j_n$ and $\beta_{|j|}$ are positive real constants). The equation (1.1) is an equation for $x = x(t)$ and is the equation of motion in $\mathbb{R}^d$ of a relativistic particle of mass $m = 1$ and charge $e = 1$ in an external electric field described by the scalar potential V (see [E] and, for example, Section 17 of [LL2]). In this equation x is the position of the particle, p is its impulse, F is the force acting on the particle, t is the time and c is the speed of light.

For the equation (1.1) the energy

$$E = c^2\sqrt{1 + \frac{|p(t)|^2}{c^2}} + V(x(t))$$

is an integral of motion. We denote by B_c the euclidean open ball whose radius is c and whose centre is 0.

Yajima [Y] studied in dimension 3 (without loss of generality for the case of dimension $d \geq 2$) the direct scattering of relativistic particle in an external electromagnetic field described by four vector $(V(x), \mathbf{A}(x))$ where the scalar potential V and the vector potential $\mathbf{A}$ are both rapidly decreasing. We recall the results of Yajima [Y] in our case.

Under the conditions (1.2), the following is valid (see [Y]): for any $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^d)$ such that

$$x(t) = v_- t + x_- + y_-(t), \tag{1.3}$$

where $\dot{y}_-(t) \to 0$, $y_-(t) \to 0$, as $t \to -\infty$; in addition for almost any $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$,

$$x(t) = v_+ t + x_+ + y_+(t), \tag{1.4}$$

where $v_+ \neq 0$, $|v_+| < c$, $v_+ = a(v_-, x_-)$, $x_+ = b(v_-, x_-)$, $\dot{y}_+(t) \to 0$, $y_+(t) \to 0$, as $t \to +\infty$.

The map $S : B_c \times \mathbb{R}^d \to B_c \times \mathbb{R}^d$ given by the formulas

$$v_+ = a(v_-, x_-),\ x_+ = b(v_-, x_-) \tag{1.5}$$

is called the scattering map for the equation (1.1); in addition, $a(v_-, x_-)$, $b(v_-, x_-)$ are called the scattering data for the equation (1.1).

By $\mathcal{D}(S)$ we denote the domain of definition of S; by $\mathcal{R}(S)$ we denote the range of S (by definition, if $(v_-, x_-) \in \mathcal{D}(S)$, then $v_- \neq 0$ and $a(v_-, x_-) \neq 0$).

Under the conditions (1.2), the map S has the following simple properties (see [Y]): for any $(v, x) \in B_c \times \mathbb{R}^d$, $(v, x) \in \mathcal{D}(S)$ if and only if $(-v, x) \in \mathcal{R}(S)$; $\mathcal{D}(S)$ is an open set of $B_c \times \mathbb{R}^d$ and $\mathrm{Mes}((B_c \times \mathbb{R}^d)\backslash\mathcal{D}(S)) = 0$ for the Lebesgue measure on $B_c \times \mathbb{R}^d$ induced by the Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$; the map $S : \mathcal{D}(S) \to \mathcal{R}(S)$ is continuous and preserves the element of volume, $a(v_-, x_-)^2 = v_-^2$.

If $V(x) \equiv 0$, then $a(v_-, x_-) = v_-$, $b(v_-, x_-) = x_-$, $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$. Therefore for $a(v_-, x_-)$, $b(v_-, x_-)$ we will use the following representation

$$\begin{aligned} a(v_-, x_-) =& v_- + a_{sc}(v_-, x_-) \\ b(v_-, x_-) =& x_- + b_{sc}(v_-, x_-) \end{aligned} \qquad (v_-, x_-) \in \mathcal{D}(S). \tag{1.6}$$

We will use the fact that, under the conditions (1.2), the map S is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$, where

$$\mathcal{M} = \{(v_-, x_-) \in B_c \times \mathbb{R}^d | v_- \neq 0, v_- x_- = 0\}.$$

Consider

$$T\mathbb{S}^{d-1} = \{(\theta, x) | \theta \in \mathbb{S}^{d-1}, x \in \mathbb{R}^d, \theta x = 0\},$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$.

Consider the X-ray transform P which maps each function f with the properties

$$f \in C(\mathbb{R}^d, \mathbb{R}^m),\ |f(x)| = O(|x|^{-\beta}),\ \text{as } |x| \to \infty,\ \text{for some } \beta > 1$$

into a function $Pf \in C(T\mathbb{S}^{d-1}, \mathbb{R}^m)$, where Pf is defined by

$$Pf(\theta, x) = \int_{-\infty}^{+\infty} f(t\theta + x) dt,\ (\theta, x) \in T\mathbb{S}^{d-1}.$$

Concerning the theory of the X-ray transform, the reader is referred to [R], [GGG], [Na] and [No].

Let

$$\text{(1.7a)}\quad \mu(c, d, \tilde{\beta}, \alpha, r_v, r_x, r) = \frac{1}{\sqrt{1 + \frac{r_v^2}{4(c^2 - r_v^2)}}} \times \frac{2^{2\alpha+6}(1 + 3\tilde{\beta}/c) d^2 \sqrt{d} \tilde{\beta} (r_v/\sqrt{2} + 1 - r)^3}{r(\alpha - 1)(r_v/\sqrt{2} - r)^4 (1 + r_x/\sqrt{2})^{\alpha-1}}$$

and let $z = z(c, d, \tilde{\beta}, \alpha, r_x, r)$, $z_1 = z_1(c, d, \beta_1, \alpha, r_x, r)$ and $z_2 = z_2(c, d, \beta_1, \alpha, r_x)$ be defined as the roots of the following equations

$$\text{(1.7b)}\quad \mu(c, d, \tilde{\beta}, \alpha, z, r_x, r) = 1,\ z \in]\sqrt{2}r, c[,$$

$$\text{(1.7c)}\quad \frac{z_1}{\sqrt{1 - \frac{z_1^2}{c^2}}} - \frac{2^{\alpha+4}\beta_1\sqrt{d}}{\alpha(z_1/\sqrt{2} - r)(r_x/\sqrt{2} + 1)^\alpha} = 0,\ z_1 \in]\sqrt{2}r, c[,$$

$$\text{(1.7d)}\quad \frac{z_2}{\sqrt{1 - \frac{z_2^2}{c^2}}} - \frac{8\beta_1\sqrt{d}}{\alpha(z_2/\sqrt{2})(1 + r_x/\sqrt{2})^\alpha} = 0,\ z_2 \in]0, c[,$$

where r_v, r_x and r are some nonnegative numbers such that $0 < r \leq 1$, $r < c/\sqrt{2}$, $\sqrt{2}r < r_v < c$, and where $\tilde{\beta} = \max(\beta_1, \beta_2)$.

The main results of the present work consist in the small angle scattering asymptotics and estimates for the scattering data a_{sc} and b_{sc} (and scattering solutions) for the equation (1.1) and in application of these asymptotics and estimates to inverse scattering for the equation (1.1) at high energies. Our main results include, in particular, Theorem 1.1 and Proposition 1.1 given below.

Theorem 1.1. *Let the conditions* (1.2) *be valid,* $\tilde\beta = \max(\beta_1, \beta_2)$, $(\theta, x) \in T\mathbb{S}^{d-1}$, *and let* r *be a positive constant such that* $0 < r \le 1$, $r < c/\sqrt{2}$. *Then*

$$PF(\theta, x) = \lim_{\substack{s\to c\\ s<c}} \frac{s}{\sqrt{1-\frac{s^2}{c^2}}} a_{sc}(s\theta, x), \tag{1.8a}$$

and, in addition,

(1.8b)

$$\left| PF(\theta,x) - \frac{s}{\sqrt{1-\frac{s^2}{c^2}}} a_{sc}(s\theta,x)\right| \le \frac{d^2\tilde\beta^2 2^{2\alpha+5} s(s/\sqrt{2}+1-r)^2}{\sqrt{1+\frac{s^2}{4(c^2-s^2)}}\alpha(\alpha-1)(s/\sqrt{2}-r)^4(1+|x|/\sqrt{2})^{2\alpha-1}}$$

for $s < c$, $s > z(c, d, \tilde\beta, \alpha, |x|, r)$, $s \ge z_1(c, d, \beta_1, \alpha, |x|, r)$;

(1.9a)

$$\int_{-\infty}^{0}\int_{-\infty}^{\tau} F(s\theta+x)dsd\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty} F(s\theta+x)dsd\tau + PV(\theta,x)\theta = \lim_{\substack{s\to c\\ s<c}} \frac{s^2}{\sqrt{1-\frac{s^2}{c^2}}} b_{sc}(s\theta,x).$$

and, in addition,

$$\left|\frac{b_{sc}(s\theta,x)}{\sqrt{1-\frac{s^2}{c^2}}} - \frac{1}{c^2}PV(\theta,x)\theta + \frac{1}{s^2}\int_0^{+\infty}\int_\tau^{+\infty} F(u\theta+x)dud\tau - \frac{1}{s^2}\int_{-\infty}^{0}\int_{-\infty}^{\tau} F(u\theta+x)dud\tau\right|$$

$$\le \sqrt{1-\frac{s^2}{c^2}}\left[C + \frac{d^3\sqrt{d}(\beta_2+3\frac{\beta_1\beta_2}{c})\beta_1 2^{3\alpha+8}(\frac{s}{\sqrt{2}}+1-r)^3}{(1-\frac{3}{4}\frac{s^2}{c^2})\alpha(\alpha-1)^2(\frac{s}{\sqrt{2}}-r)^6(1+\frac{|x|}{\sqrt{2}})^{2\alpha-2}}\right] \tag{1.9b}$$

for $s < c$, $s > z(c, d, \tilde\beta, \alpha, |x|, r)$, $s \ge \max(z_1(c, d, \beta_1, \alpha, |x|, r), z_2(c, d, \beta_1, \alpha, |x|))$ *and some constant* $C = C(c, d, \beta_0, \beta_1, \alpha, |x|)$ *which can be given explicitly.*

Consider the vector-function w of (θ, x) arising in the left-hand side of (1.9a) :

$$w(V,\theta,x) = \int_{-\infty}^{0}\int_{-\infty}^{\tau} F(s\theta+x)dsd\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty} F(s\theta+x)dsd\tau$$
$$+ PV(\theta,x)\theta,\ (\theta,x)\in T\mathbb{S}^{d-1}.$$

Proposition 1.1. *The vector* w *as a function of potential* V *satisfying the conditions* (1.2) *and of* $(\theta, x) \in T\mathbb{S}^{d-1}$ *has the following simple properties:*

1. *under the conditions* (1.2), *for any potential* V *the vector* $w(V, \theta, x)$ *is orthogonal to* θ,
2. *there exists a potential* V *which satisfies the conditions* (1.2) *and for which* $w(V, \theta, x)$

isn't null for all $(\theta, x) \in T\mathbb{S}^{d-1}$,
3. *for any spherical symmetric potential* V *satisfying the conditions* (1.2) *we have* $w(V, \theta, x) = 0$ *for all* $(\theta, x) \in T\mathbb{S}^{d-1}$.

From (1.8a) and inversion formulas for the X-ray transform for $d \geq 2$ (see [R], [GGG], [Na], [No]) it follows that a_{sc} determines uniquely F at high energies. Moreover for $d \geq 2$ methods of reconstruction of f from Pf (see [R], [GGG], [Na], [No]) permit to reconstruct F from the velocity valued component a of the scattering map at high energies. The formula (1.9a) and the item 3 of Proposition 1.1 show that the first term of the asymptotics of b_{sc} doesn't determine uniquely the potential V or the force F. The item 2 of Proposition 1.1 ensures us that the asymptotics which was found for b_{sc} is nontrivial. Note that F. Nicoleau paid our attention to the fact that, in addition of the item 3 of Proposition 1.1, $w(V, \theta, x), (\theta, x) \in T\mathbb{S}^{d-1}$, uniquely determines V satisfying (1.2) modulo spherical symmetric potentials.

Inverse scattering for the classical multidimensional Newton equation was first studied by Novikov [No] (the existence of the scattering states, asymptotic completness and scattering map for the classical Newton equation was studied by Simon [S]). Novikov proved two formulas which link scattering data at high energies to the X-ray transform of F and V. These formulas are generalized to the relativistic case by the formulas (1.8a) and (1.9a) of Theorem 1.1. Then applying results on inversion of the X-ray transform, Novikov obtains that at high energies the velocity valued component of the scattering data determines uniquely the X-ray transform of F whereas the configuration valued component of the scattering operator determines uniquely the X-ray transform of V. Note that in the relativistic case (due to the formula (1.9a) and Proposition 1.1) the asymptotics of b_{sc} doesn't determine uniquely F. We follow Novikov's framework [No] to obtain our results. Note also that for the classical multidimensional Newton equation in a bounded open strictly convex domain an inverse boundary value problem at high energies was first studied in [GN].

Further our paper is organized as follows. In Section 2 we transform the differential equation (1.1) with initial conditions (1.3) in an integral equation which takes the form $y_- = A_{v_-,x_-}(y_-)$. Then we study A_{v_-,x_-} on a suitable space and we give estimates and contraction estimates about A_{v_-,x_-} (Lemmas 2.1, 2.2, 2.3). In Section 3 we give estimates and asymptotics for the deflection $y_-(t)$ from (1.3) and for scattering data $a_{sc}(v_-, x_-), b_{sc}(v_-, x_-)$ from (1.6) (Theorem 3.1 and Theorem 3.2). From these estimates and asymptotics the formulas (1.8a) and (1.9a) will follow when the parameters $c, \beta_m,\ \alpha,\ d,\ \hat{v}_-,\ x_-$ are fixed and $|v_-|$ increases (where $\beta_{|j|},\ \alpha,\ d$ are constants from (1.2), $\beta_m = \max(\beta_0, \beta_1, \beta_2)$; $\hat{v}_- = v_-/|v_-|$). In these cases $\sup_{t\in\mathbb{R}} |\theta(t)|$ decreases, where $\theta(t)$ denotes the angle between the vectors $\dot{x}(t) = v_- + \dot{y}_-(t)$ and v_-, and we deal with small angle scattering. Note that, under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup_{t\in\mathbb{R}} |\theta(t)| < \frac{1}{4}\pi$ and we deal with rather small angle scattering (concerning the term "small angle scattering" see [No] and Section 20 of [LL1]). Theorem 1.1 follows from Theorem 3.1 and Theorem 3.2. Section 4, Section 5 and Section 6 are devoted to Proofs of our Theorems and Lemmas.

2. A contraction map

Let us transform the differential equation (1.1) in an integral equation. Consider the function $g : \mathbb{R}^d \to B_c$ defined by

$$g(x) = \frac{x}{\sqrt{1 + \frac{|x|^2}{c^2}}}$$

where $x \in \mathbb{R}^d$. One can see that g has, in particular, the following simple properties :

$$|g(x) - g(y)| \le \sqrt{d}|x - y|, \text{ for } x, y \in \mathbb{R}^d, \tag{2.1}$$

g is an infinitely smooth diffeomorphism between $\mathbb{R}^d$ and B_c, and its inverse is given by

$$\gamma(x) = \frac{x}{\sqrt{1 - \frac{|x|^2}{c^2}}}, \ x \in B_c.$$

Now, if x satisfies the differential equation (1.1) and the initial conditions (1.3), then x satisfies the integral equation

$$x(t) = v_- t + x_- + \int_{-\infty}^{t} \left[g\left(\gamma(v_-) + \int_{-\infty}^{\tau} F(x(s))ds \right) - v_- \right] d\tau, \tag{2.2}$$

where $F(x) = -\nabla V(x),\ v_- \in B_c \backslash \{0\}$.

For $y_-(t)$ this equation takes the form

$$y_-(t) = A_{v_-,x_-}(y_-)(t), \tag{2.3}$$

where $A_{v_-,x_-}(f)(t) = \int\limits_{-\infty}^{t} \left[g(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_- s + x_- + f(s))ds) - v_- \right] d\tau,\ v_- \in B_c \backslash \{0\}$.

From (2.2), (1.2), (2.1) (applied on "x"$= \gamma(v_-) + \int_{-\infty}^{\tau} F(x(s))ds$ and "y"$= \gamma(v_-)$) and $y_-(t) \in C(\mathbb{R}, \mathbb{R}^d),\ y_-(t) \to 0$, as $t \to -\infty$, it follows, in particular, that

$$y_-(t) \in C^1(\mathbb{R}, \mathbb{R}^d) \text{ and } |\dot{y}_-(t)| = O(|t|^{-\alpha}),\ |y_-(t)| = O(|t|^{-\alpha+1}), \text{ as } t \to -\infty, \tag{2.4}$$

where $v_- \in B_c \backslash \{0\}$ and x_- are fixed.

Consider the complete metric space

$$\begin{aligned} M_{T,r} =& \{ f \in C^1(] - \infty, T], \mathbb{R}^d) |\ \|f\|_T \le r \}, \\ \text{where } \|f\|_T =& \max \left(\sup_{t \in]-\infty, T]} |\dot{f}(t)|, \sup_{t \in]-\infty, T]} |f(t) - t\dot{f}(t)| \right) \end{aligned} \tag{2.5}$$

(where for $T=+\infty$ we understand $]-\infty,T]$ as $]-\infty,+\infty[$). From (2.4) it follows that, at fixed $T<+\infty$,

$$y_-(t)\in M_{T,r} \text{ for some } r \text{ depending on } y_-(t) \text{ and } T. \tag{2.6}$$

Lemma 2.1. *Under the conditions* (1.2), *the following is valid: if* $f\in M_{T,r},\ 0<r\le 1,\ r<c/\sqrt{2}$, $|v_-|<c$, $|v_-|\ge z_1(c,d,\beta_1,\alpha,|x_-|,r)$, $v_-x_-=0$, *then*

$$\|A_{v_-,x_-}(f)\|_T \le \rho_T(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)$$
$$= \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}}\frac{2^{\alpha+1}d\beta_1(|v_-|/\sqrt{2}+1-r)}{(\alpha-1)(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2}-(|v_-|/\sqrt{2}-r)T)^{\alpha-1}} \tag{2.7a}$$

for $T\le 0$,

$$\|A_{v_-,x_-}(f)\|_T \le \rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)$$
$$= \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}}\frac{2^{\alpha+2}d\beta_1(|v_-|/\sqrt{2}+1-r)}{(\alpha-1)(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2})^{\alpha-1}} \tag{2.7b}$$

for $T\le+\infty$;

if $f_1,f_2\in M_{T,r},\ 0<r\le 1,\ r<c/\sqrt{2},\ |v_-|<c,\ |v_-|\ge z_1(c,d,\beta_1,\alpha,|x_-|,r),\ v_-x_-=0$, *then*

$$\|A_{v_-,x_-}(f_2)-A_{v_-,x_-}(f_1)\|_T\le \lambda_T(c,d,\beta_2,\alpha,|v_-|,|x_-|,r)\|f_2-f_1\|_T, \tag{2.8a}$$

$$\lambda_T(c,d,\beta_2,\alpha,|v_-|,|x_-|,r) = \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}(\alpha-1)} \times\frac{2^{\alpha+2}d\sqrt{d}\beta_2(|v_-|/\sqrt{2}+1-r)^2}{(|v_-|/\sqrt{2}-r)^3(1+|x_-|/\sqrt{2}-(|v_-|/\sqrt{2}-r)T)^{\alpha-1}}$$

for $T\le 0$,

$$\|A_{v_-,x_-}(f_2)-A_{v_-,x_-}(f_1)\|_T\le \lambda(c,d,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,r)\|f_2-f_1\|_T, \tag{2.8b}$$

$$\lambda(c,d,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,r) = \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}} \times\frac{2^{2\alpha+6}(\beta_2+3\beta_1\beta_2/c)d^2\sqrt{d}(|v_-|/\sqrt{2}+1-r)^3}{(\alpha-1)(|v_-|/\sqrt{2}-r)^4(1+|x_-|/\sqrt{2})^{\alpha-1}}$$

for $T\le+\infty$.

Note that

$$\max\left(\frac{\rho_T(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)}{r},\lambda_T(c,d,\beta_2,\alpha,|v_-|,|x_-|,r)\right)$$

$$\begin{aligned}
\text{(2.9a)}\qquad &\le \mu_T(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r)\\
&= \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}}\\
&\quad\times \frac{2^{\alpha+2}d\sqrt{d}\tilde\beta(|v_-|/\sqrt{2}+1-r)^2}{r(\alpha-1)(|v_-|/\sqrt{2}-r)^3(1+|x_-|/\sqrt{2}-(|v_-|/\sqrt{2}-r)T)^{\alpha-1}}
\end{aligned}$$

for $T \le 0$,

$$\begin{aligned}
&\max\left(\frac{\rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)}{r},\lambda(c,d,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,r)\right)\\
\text{(2.9b)}\qquad &\le \mu(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r)\\
&= \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}}\\
&\quad\times \frac{2^{2\alpha+6}(1+3\tilde\beta/c)d^2\sqrt{d}\tilde\beta(|v_-|/\sqrt{2}+1-r)^3}{r(\alpha-1)(|v_-|/\sqrt{2}-r)^4(1+|x_-|/\sqrt{2})^{\alpha-1}}
\end{aligned}$$

for $T \le +\infty$, where $\tilde\beta = \max(\beta_1,\beta_2)$, $0 < r \le 1$, $r < c/\sqrt{2}$, $|v_-| < c$, $|v_-| \ge z_1(c,d,\beta_1,\alpha,|x_-|,r)$, $v_-x_- = 0$.

From Lemma 2.1 and the estimates (2.9) we obtain the following result.

Corollary 2.1. *Under the conditions* (1.2), $0 < r \le 1$, $r < c/\sqrt{2}$, $|v_-| < c$, $|v_-| \ge z_1(c,d,\beta_1,\alpha,|x_-|,r)$, $v_-x_- = 0$, *the following result is valid:*
if $\mu_T(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r) < 1$, *then* A_{v_-,x_-} *is a contraction map in* $M_{T,r}$ *for* $T \le 0$;
if $\mu(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r) < 1$, *then* A_{v_-,x_-} *is a contraction map in* $M_{T,r}$ *for* $T \le +\infty$.

Taking into account (2.6) and using Lemma 2.1, Corollary 2.1 and the lemma about the contraction maps we will study the solution $y_-(t)$ of the equation (2.3) in $M_{T,r}$.

We will use also the following results.

Lemma 2.2. *Under the conditions* (1.2), $f \in M_{T,r}$, $0 < r \le 1$, $r < c/\sqrt{2}$, $|v_-| < c$, $|v_-| \ge z_1(c,d,\beta_1,\alpha,|x_-|,r)$, $v_-x_- = 0$, *the following is valid:*

$$\begin{aligned}
|\dot A_{v_-,x_-}(f)(t)| &\le \zeta_-(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t)\\
\text{(2.10)}\qquad &= \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}}\\
&\quad\times \frac{d\beta_1 2^{\alpha+1}}{\alpha(|v_-|/\sqrt{2}-r)(1+|x_-|/\sqrt{2}-(|v_-|/\sqrt{2}-r)t)^{\alpha}},
\end{aligned}$$

$$\begin{aligned}
|A_{v_-,x_-}(f)(t)| &\le \xi_-(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t)\\
\text{(2.11)}\qquad &= \frac{1}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}}\\
&\quad\times \frac{d\beta_1 2^{\alpha+1}}{(\alpha-1)\alpha(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2}-(|v_-|/\sqrt{2}-r)t)^{\alpha-1}},
\end{aligned}$$

for $t \le T,\ T \le 0$;

$$A_{v_-,x_-}(f)(t) = k_{v_-,x_-}(f)t + l_{v_-,x_-}(f) + H_{v_-,x_-}(f)(t), \tag{2.12}$$

where

$$k_{v_-,x_-}(f) = g(\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_-s + x_- + f(s))\, ds) - v_-, \tag{2.13a}$$

$$\begin{aligned} l_{v_-,x_-}(f) = &\int_{-\infty}^{0} \left[g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_-s + x_- + f(s))\, ds) - v_- \right] d\tau \\ &+ \int_{0}^{+\infty} \Bigg[g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_-s + x_- + f(s))\, ds) \\ &- g(\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_-s + x_- + f(s))\, ds) \Bigg] d\tau, \end{aligned} \tag{2.13b}$$

$$|k_{v_-,x_-}(f)| \le 2\zeta_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, 0), \tag{2.14a}$$

$$|l_{v_-,x_-}(f)| \le 2\xi_-(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, 0), \tag{2.14b}$$

$$\begin{aligned} |\dot{H}_{v_-,x_-}(f)(t)| \le &\zeta_+(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t) \\ = &\frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}\alpha(|v_-|/\sqrt{2} - r)} \\ &\times \frac{d\beta_1 2^{\alpha+1}}{(1 + |x_-|/\sqrt{2} + (|v_-|/\sqrt{2} - r)t)^{\alpha}}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} |H_{v_-,x_-}(f)(t)| \le &\xi_+(c, d, \beta_1, \alpha, |v_-|, |x_-|, r, t) \\ = &\frac{1}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}\alpha(\alpha - 1)(|v_-|/\sqrt{2} - r)^2} \\ &\times \frac{d\beta_1 2^{\alpha+1}}{(1 + |x_-|/\sqrt{2} + (|v_-|/\sqrt{2} - r)t)^{\alpha-1}}, \end{aligned} \tag{2.16}$$

for $T = +\infty,\ t \ge 0$.

Lemma 2.3. *Let the conditions* (1.2) *be valid,* $y_-(t) \in M_{T,r}$ *be a solution of* (2.3), $T = +\infty,\ 0 < r \le 1,\ r < c/\sqrt{2}$, $|v_-| < c$, $|v_-| \ge z_1(c, d, \beta_1, \alpha, |x_-|, r),\ v_- x_- = 0$, *then*

$$\begin{aligned} |k_{v_-,x_-}(y_-) - k_{v_-,x_-}(0)| \le &\varepsilon'_a(c, d, \beta_1, \beta_2, \alpha, |v_-|, |x_-|, r) \\ = &\frac{d\sqrt{d}\beta_2 2^{\alpha+3}(|v_-|/\sqrt{2} + 1 - r)}{\alpha(|v_-|/\sqrt{2} - r)^2(1 + |x_-|/\sqrt{2})^{\alpha}} \\ &\times \frac{\rho(c, d, \beta_1, \alpha, |v_-|, |x_-|, r)}{\sqrt{1 + |v_-|^2/(4(c^2 - |v_-|^2))}}, \end{aligned} \tag{2.17a}$$

$$\left| \frac{k_{v_-,x_-}(y_-)}{\sqrt{1 - \frac{|v_-|^2}{c^2}}} - \int_{-\infty}^{+\infty} F(x_- + v_-s)\, ds \right| \le \varepsilon_a(c, d, \beta_1, \beta_2, \alpha, |v_-|, |x_-|, r)$$

$$
\text{(2.17b)} \qquad = \frac{d\beta_2 2^{\alpha+3}(1+|v_-|/\sqrt{2}-r)\rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)}{\alpha(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2})^{\alpha}},
$$

$$
|l_{v_-,x_-}(y_-) - l_{v_-,x_-}(0)| \le \varepsilon_b(c,d,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,r)
$$

$$
\text{(2.17c)} \qquad = \frac{d^2\sqrt{d}(\beta_2+3\beta_1\beta_2/c)2^{2\alpha+6}(|v_-|/\sqrt{2}+1-r)^2}{\alpha(\alpha-1)(|v_-|/\sqrt{2}-r)^4(1+|x_-|/\sqrt{2})^{\alpha-1}} \times \frac{\rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)}{\sqrt{1+|v_-|^2/(4(c^2-|v_-|^2))}}.
$$

Proofs of Lemmas 2.1, 2.2, 2.3 are given in Section 5.

3. Small angle scattering

Under the conditions (1.2), for any $(v_-, x_-) \in B_c \times \mathbb{R}^d$, $v_- \neq 0$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^d)$ with the initial conditions (1.3). Consider the function $y_-(t)$ from (1.3). This function describes deflection from free motion.

Using Corollary 2.1 the lemma about contraction maps, and Lemmas 2.2 and 2.3 we obtain the following result.

Theorem 3.1. *Let the conditions* (1.2) *be valid,* $\mu(c,d,\tilde\beta,\alpha,|v_-|,|x_-|,r) < 1$, $\tilde\beta = \max(\beta_1,\beta_2)$, $0 < r \le 1$, $r < c/\sqrt{2}$, $|v_-| < c$, $|v_-| \ge z_1(c,d,\beta_1,\alpha,|x_-|,r)$, $v_-x_- = 0$. *Then the deflection* $y_-(t)$ *has the following properties:*

$$
\text{(3.1)} \qquad y_- \in M_{T,r},\ T = +\infty;
$$

$$
\text{(3.2)} \qquad |\dot y_-(t)| \le \zeta_-(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t),
$$

$$
\text{(3.3)} \qquad |y_-(t)| \le \xi_-(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t) \ \ \textit{for} \ \ t \le 0;
$$

$$
\text{(3.4)} \qquad y_-(t) = a_{sc}(v_-,x_-)t + b_{sc}(v_-,x_-) + h(v_-,x_-,t),
$$

where

$$
\left| a_{sc}(v_-,x_-) - \left[\frac{\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_-s+x_-)ds}{\sqrt{1+\frac{|\gamma(v_-)+\int_{-\infty}^{+\infty} F(v_-s+x_-)ds|^2}{c^2}}} - v_- \right] \right|
$$

$$
\text{(3.5a)} \qquad \le \varepsilon_a'(c,d,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,r),
$$

$$
\text{(3.5b)} \qquad \left| \frac{a_{sc}(v_-,x_-)}{\sqrt{1-\frac{|v_-|^2}{c^2}}} - \int_{-\infty}^{+\infty} F(v_-s+x_-)ds \right| \le \varepsilon_a(c,d,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,r)
$$

$$
\text{(3.5c)} \qquad |b_{sc}(v_-,x_-) - l_{v_-,x_-}(0)| \le \varepsilon_b(c,d,\beta_1,\beta_2,\alpha,|v_-|,|x_-|,r),
$$

(3.6a) $$|a_{sc}(v_-,x_-)| \leq 2\zeta_-(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,0),$$

(3.6b) $$|b_{sc}(v_-,x_-)| \leq 2\xi_-(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,0),$$

(3.7) $$|\dot{h}(v_-,x_-,t)| \leq \zeta_+(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t),$$

(3.8) $$|h(v_-,x_-,t)| \leq \xi_+(c,d,\beta_1,\alpha,|v_-|,|x_-|,r,t)$$

for $t \geq 0$, *where* $l_{v_-,x_-}(0)$ *(resp.* ε'_a, ε_a, ε_b, ζ_-, ζ_+, ξ_- *and* ξ_+*) is defined in* (2.13b) (*resp.* (2.17a), (2.17b), (2.17c), (2.10), (2.15), (2.11) *and* (2.16)).

We will use the following observations.

(I) Let $0 < r \leq 1, r < c/\sqrt{2},\ 0 \leq u$

$$\frac{s_1}{\sqrt{1-\frac{s_1^2}{c^2}}} - \frac{2^{\alpha+4}\beta_1\sqrt{d}}{\alpha(s_1/\sqrt{2}-r)(u/\sqrt{2}+1)^\alpha} > \frac{s_2}{\sqrt{1-\frac{s_2^2}{c^2}}} - \frac{2^{\alpha+4}\beta_1\sqrt{d}}{\alpha(s_2/\sqrt{2}-r)(u/\sqrt{2}+1)^\alpha}$$

for $\sqrt{2}r < s_2 < s_1 < c$.

(II) Let $0 < r \leq 1, r < c/\sqrt{2},\ u \in]\sqrt{2}r, c[$,

$$\frac{u}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{2^{\alpha+4}\beta_1\sqrt{d}}{\alpha(u/\sqrt{2}-r)(s_1/\sqrt{2}+1)^\alpha} > \frac{u}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{2^{\alpha+4}\beta_1\sqrt{d}}{\alpha(u/\sqrt{2}-r)(s_2/\sqrt{2}+1)^\alpha}$$

for $0 \leq s_2 < s_1$.

(III) Let $0 < r \leq 1,\ r < c/\sqrt{2}$, x some real nonnegative number, $\tilde{\beta} = \max(\beta_1, \beta_2)$ and $\sqrt{2}r < s < c$ then

$$\mu(c,d,\tilde{\beta},\alpha,s,|x|,r) < 1 \Leftrightarrow s > z(c,d,\tilde{\beta},\alpha,|x|,r).$$

Observations (I) and (II) imply that $z_1(c,d,\beta_1,\alpha,s_2,r) > z_1(c,d,\beta_1,\alpha,s_1,r)$ for $\sqrt{2}r < s_2 < s_1 < c$ when $c,\ \beta_1,\ \alpha,\ d,\ r$ are fixed.

Theorem 3.1 gives, in particular, estimates for the scattering process and asymptotics for the velocity valued component of the scattering map when $c,\ \beta_1,\ \beta_2,\ \alpha,\ d,\ \hat{v}_-,\ x_-$ are fixed (where $\hat{v}_- = v_-/|v_-|$) and $|v_-|$ increases or, e.g., $c,\ \beta_1,\ \beta_2,\ \alpha,\ d,\ v_-,\ \hat{x}_-$ are fixed and $|x_-|$ increases. In these cases $\sup_{t\in\mathbb{R}}|\theta(t)|$ decreases, where $\theta(t)$ denotes the angle between the vectors $\dot{x}(t) = v_- + \dot{y}_-(t)$ and v_-, and we deal with small angle scattering. Note that already under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup_{t\in\mathbb{R}}|\theta(t)| < \frac{1}{4}\pi$ and we deal with a rather small angle scattering. Theorem 3.1 with (3.5c) will give the asymptotics of the configuration valued component $b(v_-,x_-)$ of the scattering map if we can study the asymptotics of $l_{v_-,x_-}(0)$. This is the subject of Theorem 3.2.

Theorem 3.2. *Let* $c,\ d,\ \beta_0,\ \beta_1,\ \alpha,\ |x|$ *be fixed. Then there exists a constant* $C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ *such that*

$$\left|\frac{l_{v,x}(0)}{\sqrt{1-\frac{|v|^2}{c^2}}} - \frac{1}{c^2}PV(\hat{v},x)\hat{v} + \frac{1}{|v|^2}\int_0^{+\infty}\int_\tau^{+\infty} F(u\hat{v}+x)dud\tau - \frac{1}{|v|^2}\int_{-\infty}^0\int_{-\infty}^\tau F(u\hat{v}+x)dud\tau\right|$$

(3.9) $$\leq C_{c,d,\beta_0,\beta_1,\alpha,|x|}\sqrt{1-\frac{|v|^2}{c^2}}$$

for any $v \in B_c$, $|v| \geq z_2(c,d,\beta_1,\alpha,|x|)$, $vx = 0$, *and where* $\hat{v} = v/|v|$.

The proof of Theorem 3.2 is given in Section 6. Using this proof one can compute $C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ explicitly.

4. Preliminaries for the main proofs

4.1 Inequalities for F.

Lemma 4.1. *Under the conditions* (1.2), *the following estimates are valid:*

$$|F(x)| = \left(\sum_{j=1}^{d} |\frac{\partial}{\partial x_j} V(x)|^2\right)^{\frac{1}{2}} \leq \beta_1 \sqrt{d}(1+|x|)^{-(\alpha+1)} \text{ for } x \in \mathbb{R}^d, \tag{4.1}$$

$$|F(x) - F(y)| \leq \beta_2 d \sup_{\varepsilon\in[0,1]} (1+|\varepsilon x + (1-\varepsilon)y|)^{-(\alpha+2)}|x-y|, \text{ for } x,\ y \in \mathbb{R}^d. \tag{4.2}$$

Lemma 4.1 follows directly from the formula $F(x) = -\nabla V(x)$ and the conditions (1.2).

4.2 Infinitely smooth function $g : \mathbb{R}^d \to B_c$.

Lemma 4.2. *The following estimates hold:*

$$|\nabla g_i(x)|^2 \leq \frac{1}{1+\frac{|x|^2}{c^2}} \text{ for } x \in \mathbb{R}^d,\ i = 1..d, \tag{4.3}$$

$$|g(x) - g(y)| \leq \sqrt{d} \sup_{\varepsilon\in[0,1]} \frac{1}{\sqrt{1+\frac{|\varepsilon x+(1-\varepsilon)y|^2}{c^2}}}|x-y|, \text{ for } x,\ y \in \mathbb{R}^d, \tag{4.4}$$

$$|\nabla g_j(x) - \nabla g_j(y)| \leq \frac{3\sqrt{d}}{c} \sup_{\varepsilon\in[0,1]} \frac{1}{1+\frac{|\varepsilon x+(1-\varepsilon)y|^2}{c^2}}|x-y|, \text{ for } x,\ y \in \mathbb{R}^d. \tag{4.5}$$

where $g = (g_1,..,g_d)$.

Lemma 4.2 follows from straighforward calculations.

Remark 4.1. Using the growth properties of $g(p)$ with respect to $|p|$ and following Novikov's framework [No], we will easily generalize some of the results of [No] to the relativistic case. Note that $\frac{1}{1+|p|^2/c^2} \to 0$ when $p \in \mathbb{R}^d$, $|p| \to +\infty$.

4.3 Some estimates of integrals.

We will use the following estimates. For $a > 0$, $b > 0$, $\beta > 1$,

$$\int_{-\infty}^{t} (a+b|s|)^{-\beta}\, ds = \frac{1}{(\beta-1)b(a-bt)^{\beta-1}}, \text{ for } t \leq 0, \tag{4.6}$$

$$\int_{-\infty}^{t} (a+b|s|)^{-\beta}\, ds \leq \frac{2}{(\beta-1)ba^{\beta-1}}, \text{ for } t \geq 0. \tag{4.7}$$

For $a>0,\ b>0,\ \beta>2$,

$$\int_{-\infty}^{t}\int_{-\infty}^{\tau}(a+b|s|)^{-\beta}\,ds\,d\tau = \frac{1}{(\beta-2)(\beta-1)b^2(a-bt)^{\beta-2}},\ \text{for } t\le 0, \tag{4.8}$$

$$\int_{0}^{t}\int_{\tau}^{t}(a+bs)^{-\beta}\,ds\ d\tau \le \frac{1}{(\beta-2)(\beta-1)b^2a^{\beta-2}},\ \text{for } t\ge 0. \tag{4.9}$$

For $a\ge 1,\ b>0,\ \beta>2$,

$$\int_{-\infty}^{t}(a+b|s|)^{-\beta}(1+|s|)\ ds \le \frac{b+1}{(\beta-2)b^2(a-bt)^{\beta-2}},\ \text{for } t\le 0, \tag{4.10}$$

$$\int_{-\infty}^{t}(a+b|s|)^{-\beta}(1+|s|)\ ds \le 2\frac{b+1}{(\beta-2)b^2a^{\beta-2}},\ \text{for } t\ge 0. \tag{4.11}$$

For $a\ge 1,\ b>0,\ \beta>3$,

$$\int_{0}^{t}\int_{\tau}^{t}(a+bs)^{-\beta}(1+s)\ ds\ d\tau \le \frac{b+1}{(\beta-3)(\beta-2)b^3a^{\beta-3}},\ \text{for } t\ge 0. \tag{4.12}$$

For the proof of (4.6)-(4.12), see [No].

4.4 About $z_1(c,d,\beta_1,\alpha,|x_-|,r)$.

Let $c,\ d,\ \beta_1,\ \alpha,|x_-|,\ 0<r\le 1,\ r<c/\sqrt{2}$, be fixed. We consider the one-dimensional infinitely smooth function $\sigma:]\sqrt{2}r,c[\to\mathbb{R}$ defined by

$$\sigma(s)=\frac{s}{\sqrt{1-\frac{s^2}{c^2}}}-\frac{2^{\alpha+4}\beta_1\sqrt{d}}{\alpha(s/\sqrt{2}-r)(|x_-|/\sqrt{2}+1)^{\alpha}}.$$

σ is an increasing function (its derivative is a positive function) and as a consequence $z_1(c,d,\beta_1,\alpha,|x_-|,r)$ is well defined in Introduction and the observation (I) of Section 3 holds.

4.5 About $M_{T,r},\ 0<r\le 1,\ r<c/\sqrt{2}$.

Lemma 4.3. *Let* $f,\ f_1,\ f_2\in M_{T,r},\ v_-\in B_c\backslash\{0\},\ v_-x_-=0,\ |v_-|>\sqrt{2}r$, *then*

$$\varepsilon f_1+(1-\varepsilon)f_2\in M_{T,r},\ \textit{for}\ 0\le\varepsilon\le 1, \tag{4.13}$$

$$2(1+|x_-+v_-s+f(s)|)\ge(1+|x_-|/\sqrt{2}+(|v_-|/\sqrt{2}-r)|s|),\ \textit{for}\ s\le T, \tag{4.14}$$

(4.15)
$$\left|\int_{-\infty}^{t} F(v_-s+x_-+f(s))ds\right| \le \frac{\beta_1\sqrt{d}2^{\alpha+2}}{\alpha(|v_-|/\sqrt{2}-r)(|x_-|/\sqrt{2}+1)^{\alpha}},\ \textit{for}\ t\in]-\infty,+\infty],$$

$$\left(1+\frac{1}{c^2}\left|\gamma(v_-)+\varepsilon_1\int_{-\infty}^{t} F(v_-s+x_-+f_1(s))ds+\varepsilon_2\int_{w}^{u} F(v_-s+x_-+f_2(s))ds\right|^2\right)^{-\beta}$$
$$\le (1+\frac{|v_-|^2}{4(c^2-|v_-|^2)})^{-\beta}, \tag{4.16}$$

for $u,t\in]-\infty,T]$, $w\in[-\infty,u]$, $\beta>0$, $-1\le\varepsilon_1,\varepsilon_2\le 1, f_1,f_2\in M_{T,r}$ *and if* $|v_-|\ge z_1(c,d,\beta_1,\alpha,|x_-|,r)$, $|v_-|<c$, *where* γ *is defined by*

$$\gamma(v)=\frac{v}{\sqrt{1-|v|^2/c^2}},$$

for $v\in B_c$.

Proof of Lemma 4.3. For the proof of (4.14) see [No]. Inequality (4.1) with (4.14) and (4.7) proves (4.15). (4.13) follows from the definition of $M_{T,r}$. Inequality (4.15) gives in particular for $u,t\in]-\infty,T],\ w\in[-\infty,u],\ \beta>0,\ -1\le\varepsilon_1,\varepsilon_2\le 1, f_1,f_2\in M_{T,r}$

$$\begin{aligned}
&|\gamma(v_-)+\varepsilon_1\int_{-\infty}^{t} F(v_-s+x_-+f_1(s))ds+\varepsilon_2\int_{w}^{u} F(v_-s+x_-+f_2(s))ds|\\
&\ge |\gamma(v_-)|-\frac{\beta_1\sqrt{d}2^{\alpha+3}}{\alpha(|v_-|/\sqrt{2}-r)(|x_-|/\sqrt{2}+1)^{\alpha}}\\
&=\frac{|v_-|}{\sqrt{1-|v_-|^2/c^2}}-\frac{\beta_1\sqrt{d}2^{\alpha+3}}{\alpha(|v_-|/\sqrt{2}-r)(|x_-|/\sqrt{2}+1)^{\alpha}}\\
&\ge\frac{c|v_-|}{2\sqrt{c^2-|v_-|^2}},\ \text{if}\ |v_-|\ge z_1(c,d,\beta_1,\alpha,|x_-|,r),\ |v_-|<c,
\end{aligned}$$

which implies (4.16).

5. Proofs of Lemmas 2.1, 2.2, 2.3

Proof of Lemma 2.1. The property

$$A_{v_-,x_-}(f)\in C^1(]-\infty,T],\mathbb{R}^d)\ \text{for}\ f\in M_{T,r}\ (0<r\le 1,\ r<|v_-|/\sqrt{2}) \tag{5.1}$$

follows from (1.2), (2.1) (applied on "x"$=\gamma(v_-)+\int_{-\infty}^{\tau}F(v_-s+x_-+f(s))ds$ and "y"$=\gamma(v_-)$) and the definition of $A_{v_-,x_-}(f)$.

Now we always suppose that $0 < r \le 1$, $r < c/\sqrt{2}$, $|v_-| \ge z_1(c, d, \beta_1, \alpha, |x_-|, r)$, $|v_-| < c$, $v_- x_- = 0$. Consider

(5.2)
$$A_{v_-,x_-}(f)(t) = \int\limits_{-\infty}^{t} \left[g(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_- s + x_- + f(s))ds) - v_- \right] d\tau$$
for $f \in M_{T,r}$.
$$\frac{d}{dt} A_{v_-,x_-}(f)(t) = g(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f(s))ds) - v_-.$$

First we shall prove some estimates about $\frac{d}{dt} A_{v_-,x_-}(f)$.

Note that $g(\gamma(v_-)) = v_-$. From (5.2), (4.4) (applied on "x"$= \gamma(v_-) + \int_{-\infty}^{t} F(v_- s + x_- + f(s))ds$ and "y"$= \gamma(v_-)$), (4.1), (4.14) and (4.16) it follows that

$$(5.3)\quad |\frac{d}{dt} A_{v_-,x_-}(f)(t)| \le \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}}} \int_{-\infty}^{t} (1 + |x_-|/\sqrt{2} + (|v_-|/\sqrt{2} - r)|s|))^{-(\alpha+1)} ds.$$

Our next purpose is to prove estimates (5.5) and (5.9) given below.

From (5.3) and (4.8) and (4.6) it follows that

$$(5.4a)\quad |A_{v_-,x_-}(f)(t)| \le \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}} \alpha(\alpha-1)(\frac{|v_-|}{\sqrt{2}} - r)^2 (1 + \frac{|x_-|}{\sqrt{2}} - (\frac{|v_-|}{\sqrt{2}} - r)t)^{\alpha-1}},$$

$$(5.4b)\quad |t \frac{d}{dt} A_{v_-,x_-}(f)(t)| \le \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}} \alpha(\frac{|v_-|}{\sqrt{2}} - r)^2 (1 + \frac{|x_-|}{\sqrt{2}} - (\frac{|v_-|}{\sqrt{2}} - r)t)^{\alpha-1}},$$

for $t \le T,\ t \le 0$. From (5.4) it follows that

$$|A_{v_-,x_-}(f)(t) - t \frac{d}{dt} A_{v_-,x_-}(f)(t)|$$
$$(5.5)\qquad \le \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}} (\alpha-1)(|v_-|/\sqrt{2} - r)^2 (1 + |x_-|/\sqrt{2} - (|v_-|/\sqrt{2} - r)t)^{\alpha-1}},$$

for $t \le T,\ t \le 0$.

For $t \le T,\ t \ge 0$, note that

$$A_{v_-,x_-}(f)(t) - t \frac{d}{dt} A_{v_-,x_-}(f)(t)$$
$$(5.6)\qquad = A_{v_-,x_-}(f)(0) + \int\limits_{0}^{t} \left[g(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_- s + x_- + f(s))ds) \right.$$
$$\left. - g(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f(s))ds) \right] d\tau.$$

For $A_{v_-,x_-}(f)(0)$ we use the estimate (5.4a), i.e.

(5.7)
$$|A_{v_-,x_-}(f)(0)| \le \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1+(|v_-|^2/(4(c^2-|v_-|^2)))}\alpha(\alpha-1)(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2})^{\alpha-1}}.$$

We estimate the second term on the right-hand side of (5.6) in the following way: from (4.4), (4.16), (4.1), (4.14) and (4.9), it follows that

$$\left|\int\limits_0^t \left[g(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_-s+x_-+f(s))ds) - g(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_-s+x_-+f(s))ds) \right] d\tau \right|$$

$$\le \frac{\sqrt{d}}{\sqrt{1+\frac{|v_-|^2}{4(c^2-|v_-|^2)}}} \int_0^t \left| \int_t^\tau F(v_-s+x_-+f(s))ds \right| d\tau$$

$$\le \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1+(|v_-|^2/(4(c^2-|v_-|^2)))}\alpha(\alpha-1)(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2})^{\alpha-1}}, \tag{5.8}$$

for $0 \le t \le T$. From (5.6), (5.7) and (5.8) it follows that

$$|A_{v_-,x_-}(f)(t) - t\frac{d}{dt}A_{v_-,x_-}(f)(t)|$$
$$\le \frac{d\beta_1 2^{\alpha+2}}{\sqrt{1+\frac{|v_-|^2}{(4(c^2-|v_-|^2)}}\alpha(\alpha-1)(|v_-|/\sqrt{2}-r)^2(1+|x_-|/\sqrt{2})^{\alpha-1}} \tag{5.9}$$

for $0 \le t \le T$. Using (5.3) and (4.6) and using (5.5) we obtain (2.7a). Using (5.3) and (4.7) and using (5.9) we obtain (2.7b).

Our next purpose is to prove estimate (5.14) given below. Consider $\frac{d}{dt}(A_{v_-,x_-}(f_2)(t) - A_{v_-,x_-}(f_1)(t))$ for $f_1,\ f_2 \in M_{T,r}$ ($0 < r \le 1$, $r < c/\sqrt{2}$, $|v_-| < c$, $v_-x_- = 0$, $|v_-| \ge z_1(c,d,\beta_1,\alpha,|x_-|,r)$). First

$$\frac{d}{dt}A_{v_-,x_-}(f_2)(t) - \frac{d}{dt}A_{v_-,x_-}(f_1)(t)$$
$$= g(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_-s+x_-+f_2(s))ds) - g(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_-s+x_-+f_1(s))ds) \tag{5.10}$$

for $t \le T$. From (5.10), (4.4) and (4.16) it follows that

$$|\frac{d}{dt}(A_{v_-,x_-}(f_2)(t) - \frac{d}{dt}A_{v_-,x_-}(f_1)(t))|$$
$$\le \frac{\sqrt{d}}{\sqrt{1+(|v_-|^2/(4(c^2-|v_-|^2)))}} \int\limits_{-\infty}^{t} |F(v_-s+x_-+f_2(s)) - F(v_-s+x_-+f_1(s))|ds, \tag{5.11}$$

for $t \le T$. From (4.13), (4.14) and (4.2), it follows that

$$|F(v_-s + x_- + f_2(s)) - F(v_-s + x_- + f_1(s))| \le d\beta_2 2^{\alpha+2}(1 + |x_-|/\sqrt{2} + (|v_-|/\sqrt{2} - r)|s|)^{-(\alpha+2)}|f_2(s) - f_1(s)|, \text{ for } s \le T. \tag{5.12}$$

Moreover

$$|f_2(s) - f_1(s)| \le (1 + |s|)\|f_2 - f_1\|_T, \text{ for } s \le T. \tag{5.13}$$

Thus, from (5.11), (5.12) and (5.13) it follows that

$$|\frac{d}{dt}A_{v_-,x_-}(f_2)(t) - \frac{d}{dt}A_{v_-,x_-}(f_1)(t)| \le \frac{d\sqrt{d}\beta_2 2^{\alpha+2}\|f_2 - f_1\|_T}{\sqrt{1 + (|v_-|^2/(4(c^2 - |v_-|^2)))}} \int_{-\infty}^{t} (1 + |x_-|/\sqrt{2} + (|v_-|/\sqrt{2} - r)|s|)^{-(\alpha+2)}(1 + |s|)ds. \tag{5.14}$$

Our next purpose is to prove estimates (5.17) and (5.31) given below. From (5.14) and (4.10) and (4.6) it follows that

$$|A_{v_-,x_-}(f_2)(t) - A_{v_-,x_-}(f_1)(t)| \le \frac{d\sqrt{d}\beta_2 2^{\alpha+2}(|v_-|/\sqrt{2} + 1 - r)\|f_2 - f_1\|_T}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}}\alpha(\alpha - 1)(|v_-|/\sqrt{2} - r)^3(1 + |x_-|/\sqrt{2} - (|v_-|/\sqrt{2} - r)t)^{\alpha-1}} \tag{5.15}$$

for $t \le T,\ t \le 0$. From (5.14) and (4.10) it also follows that

$$|t||\frac{d}{dt}A_{v_-,x_-}(f_2)(t) - \frac{d}{dt}A_{v_-,x_-}(f_1)(t)| \le \frac{d\sqrt{d}\beta_2 2^{\alpha+2}(|v_-|/\sqrt{2} + 1 - r)\|f_2 - f_1\|_T}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}}\alpha(|v_-|/\sqrt{2} - r)^3(1 + |x_-|/\sqrt{2} - (|v_-|/\sqrt{2} - r)t)^{\alpha-1}} \tag{5.16}$$

for $t \le T,\ t \le 0$. Hence from (5.15) and (5.16) it follows that

$$|A_{v_-,x_-}(f_2)(t) - A_{v_-,x_-}(f_1)(t) - t\frac{d}{dt}(A_{v_-,x_-}(f_2)(t) - A_{v_-,x_-}(f_1)(t))| \le \frac{d\sqrt{d}\beta_2 2^{\alpha+2}(|v_-|/\sqrt{2} + 1 - r)\|f_2 - f_1\|_T}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}}(\alpha - 1)(|v_-|/\sqrt{2} - r)^3(1 + |x_-|/\sqrt{2} - (|v_-|/\sqrt{2} - r)t)^{\alpha-1}} \tag{5.17}$$

for $t \le T, t \le 0$.

For $0 \le t \le T$, using (5.6) we obtain

$$|A_{v_-,x_-}(f_2)(t) - A_{v_-,x_-}(f_1)(t) - t(\frac{d}{dt}A_{v_-,x_-}(f_2)(t) - \frac{d}{dt}A_{v_-,x_-}(f_1)(t))|$$

(5.18)

$$\le |A_{v_-,x_-}(f_2)(0) - A_{v_-,x_-}(f_1)(0)|$$
$$+ \left| \int\limits_0^t \left[g(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_- s + x_- + f_2(s))ds) - g(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f_2(s))ds) \right.\right.$$
$$\left.\left. - g(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_- s + x_- + f_1(s))ds) + g(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f_1(s))ds) \right] d\tau \right|.$$

From (5.15) it follows that

(5.19)

$$|A_{v_-,x_-}(f_2)(0) - A_{v_-,x_-}(f_1)(0)| \le \frac{d\sqrt{d}\beta_2 2^{\alpha+2}(|v_-|/\sqrt{2} + 1 - r)\|f_2 - f_1\|_T}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}}\alpha(\alpha-1)(\frac{|v_-|}{\sqrt{2}} - r)^3(1 + \frac{|x_-|}{\sqrt{2}})^{\alpha-1}}.$$

In order to estimate the second term of the right-hand side of (5.18), we will estimate

$$\int\limits_0^t \left[g_j(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_- s + x_- + f_2(s))ds) - g_j(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f_2(s))ds) \right.$$

(5.20)

$$\left. - g_j(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_- s + x_- + f_1(s))ds) + g_j(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f_1(s))ds) \right] d\tau$$

for $1 \le j \le d$ and $0 \le t \le T$.

Let $1 \le j \le d$ and $0 \le t \le T,\ 0 \le \tau \le t$. Note that

$$g_j(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_- s + x_- + f_2(s))ds) - g_j(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f_2(s))ds)$$

$$- \left(g_j(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_- s + x_- + f_1(s))ds) - g_j(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f_1(s))ds) \right)$$

(5.21)
$$= \Delta^1_{j,t}(\tau) + \Delta^2_{j,t}(\tau)$$

where

(5.22a)
$$\Delta^1_{j,t}(\tau) = \int_t^{\tau} (F(v_- s + x_- + f_2(s)) - F(v_- s + x_- + f_1(s)))ds$$

$$\star \int_0^1 \nabla g_j \left(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f_2(s))ds + \varepsilon \int\limits_{t}^{\tau} F(v_- s + x_- + f_2(s))ds \right) d\varepsilon,$$

(5.22b)
$$\Delta^2_{j,t}(\tau) = \int_t^\tau F(v_- s + x_- + f_1(s))ds$$

$$\star \int_0^1 \left[\nabla g_j(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f_2(s))ds + \varepsilon \int\limits_{t}^{\tau} F(v_- s + x_- + f_2(s))ds) \right.$$

$$\left. -\nabla g_j(\gamma(v_-) + \int\limits_{-\infty}^{t} F(v_- s + x_- + f_1(s))ds + \varepsilon \int\limits_{t}^{\tau} F(v_- s + x_- + f_1(s))ds) \right] d\varepsilon.$$

Using (5.22a), (4.3), (4.2), (4.16), (4.14) and (5.13), we obtain

(5.23)
$$|\Delta^1_{j,t}(\tau)| \leq \frac{d\beta_2 2^{\alpha+2}}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}}} \int_\tau^t (1 + |x_-|/\sqrt{2} + (|v_-|/\sqrt{2} - r)s)^{-(\alpha+2)}(1+s)ds \|f_2 - f_1\|_T.$$

Thus from (4.12) it follows that

(5.24)
$$\int_0^t |\Delta^1_{j,t}(\tau)| d\tau \leq \frac{d\beta_2 2^{\alpha+2}(|v_-|/\sqrt{2} + 1 - r)}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}}\alpha(\alpha-1)(|v_-|/\sqrt{2} - r)^3(1 + |x_-|/\sqrt{2})^{\alpha-1}} \|f_2 - f_1\|_T.$$

Using (5.22b), (4.5) and (4.16), we obtain

(5.25)
$$\begin{aligned} |\Delta^2_{j,t}(\tau)| \leq \int_\tau^t |F(v_- s + x_- + f_1(s))| ds & \left[\frac{3\sqrt{d}}{c(1 + (|v_-|^2/(4(c^2 - |v_-|^2))))} \right. \\ & \times \int_0^1 \left| \int_{-\infty}^t (F(v_- s + x_- + f_2(s)) - F(v_- s + x_- + f_1(s)))ds \right. \\ & \left. \left. + \varepsilon \int_t^\tau (F(v_- s + x_- + f_2(s)) - F(v_- s + x_- + f_1(s)))ds \right| d\varepsilon \right]. \end{aligned}$$

We shall use

(5.26)
$$\begin{aligned} \Big| \int_{-\infty}^t & (F(v_- s + x_- + f_2(s)) - F(v_- s + x_- + f_1(s)))ds \\ + \varepsilon \int_t^\tau & (F(v_- s + x_- + f_2(s)) - F(v_- s + x_- + f_1(s)))ds| \\ & \leq 2 \int_{-\infty}^t |(F(v_- s + x_- + f_2(s)) - F(v_- s + x_- + f_1(s)))| ds, \end{aligned}$$

for all $0 \le \varepsilon \le 1$ (we remind that $\tau \le t$).

From (5.25) and (5.26) it follows that

$$|\Delta^2_{j,t}(\tau)| \le \int_\tau^t |F(v_-s + x_- + f_1(s))| ds$$

$$\times \frac{6\sqrt{d}}{c(1 + \frac{|v_-|^2}{4(c^2-|v_-|^2)})} \int_{-\infty}^t |(F(v_-s + x_- + f_2(s)) - F(v_-s + x_- + f_1(s)))| ds. \tag{5.27}$$

Using (4.2), (4.14), (5.13) and (4.11) we obtain

$$\int_{-\infty}^t |F(v_-s+x_- + f_2(s)) - F(v_-s + x_- + f_1(s))| ds$$

$$\le \frac{d\beta_2 2^{\alpha+3}(|v_-|/\sqrt{2} + 1 - r)}{\alpha(|v_-|/\sqrt{2} - r)^2(1 + |x_-|/\sqrt{2})^\alpha} \|f_2 - f_1\|_T. \tag{5.28}$$

Using (4.1), (4.14) and (4.9) we obtain

$$\int_0^t \int_\tau^t |F(v_-s + x_- + f_1(s))| ds d\tau \le \frac{\sqrt{d}\beta_1 2^{\alpha+1}}{\alpha(\alpha-1)(|v_-|/\sqrt{2} - r)^2(1 + |x_-|/\sqrt{2})^{\alpha-1}}. \tag{5.29}$$

From (5.27), (5.28) and (5.29) it follows that

$$\int_0^t |\Delta^2_{j,t}(\tau)| d\tau \le \frac{3}{c(1 + (|v_-|^2/(4(c^2 - |v_-|^2))))}$$

$$\times \frac{d^2\beta_1\beta_2 2^{2\alpha+5}(|v_-|/\sqrt{2} + 1 - r)}{\alpha^2(\alpha-1)(|v_-|/\sqrt{2} - r)^4(1 + |x_-|/\sqrt{2})^{2\alpha-1}} \|f_2 - f_1\|_T. \tag{5.30}$$

From (5.18), (5.19), (5.21), (5.24) and (5.30), it follows that

$$|A_{v_-,x_-}(f_2)(t) - A_{v_-,x_-}(f_1)(t) - t(\frac{d}{dt}A_{v_-,x_-}(f_2)(t) - \frac{d}{dt}A_{v_-,x_-}(f_1)(t))|$$

$$\le \frac{d\sqrt{d}\beta_2 2^{\alpha+2}(|v_-|/\sqrt{2} + 1 - r)}{\sqrt{1 + (|v_-|^2/(4(c^2 - |v_-|^2)))}\alpha(\alpha-1)(|v_-|/\sqrt{2} - r)^3(1 + |x_-|/\sqrt{2})^{\alpha-1}}$$

$$\times \left[2 + \frac{3}{c\sqrt{(1 + (|v_-|^2/(4(c^2 - |v_-|^2))))}} \frac{d\beta_1 2^{\alpha+3}}{\alpha(|v_-|/\sqrt{2} - r)(1 + |x_-|/\sqrt{2})^\alpha}\right]$$

$$\times \|f_1 - f_2\|_T. \tag{5.31}$$

Using (5.14) and (4.10) and (5.17) we obtain (2.8a). Using (5.14) and (4.11) and (5.31) we obtain (2.8b).

Lemma 2.1 is proved.

Proof of Lemma 2.2. The estimates (2.10) and (2.11) follow immediately from (5.3) and (4.6) and (5.4a). From (4.1), (4.14) and (4.7) it follows that

$$\int_{-\infty}^{+\infty} F(v_- s + x_- + f(s))ds$$

converges absolutely for any $f \in M_{T,r}$. Moreover, using (4.4), (4.16) and then (4.1), (4.14) and (4.8) we obtain for $u > 0$

$$\int_u^{+\infty} \left| g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_- + f(s))ds) - g(\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_- s + x_- + f(s))ds) \right| d\tau$$

$$\leq \frac{d\beta_1 2^{\alpha+1}}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}}} \int_u^{+\infty} \int_\tau^{+\infty} (1 + |x_-|/\sqrt{2} + (|v_-|/\sqrt{2} - r)s)^{-(\alpha+1)} ds d\tau$$

$$\leq \frac{d\beta_1 2^{\alpha+1} (1 + |x_-|/\sqrt{2} + (|v_-|/\sqrt{2} - r)u)^{-(\alpha-1)}}{\sqrt{1 + \frac{|v_-|^2}{4(c^2 - |v_-|^2)}} \alpha(\alpha - 1)(|v_-|/\sqrt{2} - r)^2}. \tag{5.32}$$

As a consequence we can write

$$\begin{aligned} A_{v_-,x_-}(f)(t) = & t \left[g(\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_- s + x_- + f(s))ds) - v_- \right] \\ & + \int_{-\infty}^{0} \left[g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_- + f(s))ds) - v_- \right] d\tau \\ & + \int_{0}^{+\infty} \left[g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_- + f(s))ds) - g(\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_- s + x_- + f(s))ds) \right] d\tau \\ & - \int_{t}^{+\infty} \left[g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_- + f(s))ds) - g(\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_- s + x_- + f(s))ds) \right] d\tau \end{aligned} \tag{5.33}$$

and (2.12) and (2.13) follow, where

$$\begin{aligned} H_{v_-,x_-}(f)(t) = \int_t^{+\infty} \Bigg[& g(\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_- s + x_- + f(s))ds) \\ & - g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_- + f(s))ds) \Bigg] d\tau. \end{aligned} \tag{5.34}$$

The formulas (5.34) and (5.32) prove (2.16). Using (5.34), (4.4), (4.16), (4.1), (4.14) and (4.6), we obtain (2.15).

Using (2.13a), (4.4), (4.16), (4.1), (4.14) and (4.7) we obtain (2.14a).

We write

$$
\begin{aligned}
l_{v_-,x_-}(f) =& A_{v_-,x_-}(f)(0) + \int\limits_0^{+\infty} \Bigg[g(\gamma(v_-) + \int\limits_{-\infty}^{\tau} F(v_- s + x_- + f(s))ds) \\
& -g(\gamma(v_-) + \int\limits_{-\infty}^{+\infty} F(v_- s + x_- + f(s))ds) \Bigg] d\tau.
\end{aligned} \tag{5.35}
$$

Using (5.35), (5.7) and (5.32), we obtain (2.14b).

Thus Lemma 2.2 is proved.

Proof of Lemma 2.3. Using (2.3) and (2.7b) we obtain

$$\|y_- - 0\|_T = \|y_-\|_T \le \rho(c, d, \beta_1, \alpha, |v_-|, |x_-|, r),\ T = +\infty.$$

Using (2.13a), (5.10) with (5.14) and (4.11) ($T = +\infty$ and $t \to +\infty$), we obtain (2.17a).

From (5.35) it follows that

$$|l_{v_-,x_-}(y_-) - l_{v_-,x_-}(0)| \le |A_{v_-,x_-}(y_-)(0) - A_{v_-,x_-}(0)(0)| \tag{5.36}$$

$$
\begin{aligned}
&+ \Bigg| \lim_{t\to+\infty} \Bigg\{ \int_0^t \Bigg[g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_- + y_-(s))ds) - g(\gamma(v_-) + \int_{-\infty}^{t} F(v_- s + x_- + y_-(s))ds) \\
&- g(\gamma(v_-) + \int_{-\infty}^{\tau} F(v_- s + x_-)ds) + g(\gamma(v_-) + \int_{-\infty}^{t} F(v_- s + x_-)ds) \\
&- \Bigg(g(\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_- s + x_- + y_-(s))ds) - g(\gamma(v_-) + \int_{-\infty}^{t} F(v_- s + x_- + y_-(s))ds) \Bigg) \\
&+ \Bigg(g(\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_- s + x_-)ds) - g(\gamma(v_-) + \int_{-\infty}^{t} F(v_- s + x_-)ds) \Bigg) \Bigg] d\tau \Bigg\} \Bigg|.
\end{aligned}
$$

Using (4.4), (4.16), (4.1), (4.14) and (4.6) we obtain

$$
\begin{aligned}
t \Bigg| g(\gamma(v_-) + \int_{-\infty}^{+\infty} F(v_- s + x_- + f(s))ds) \ -g(\gamma(v_-) + \int_{-\infty}^{t} F(v_- s + x_- + f(s))ds) \Bigg| \\
\to 0 \text{ as } t \to +\infty
\end{aligned} \tag{5.37}
$$

for $f \in M_{T,r}$.

From (5.36) and (5.37) it follows that

$$|l_{v_-,x_-}(y_-) - l_{v_-,x_-}(0)| \le |A_{v_-,x_-}(y_-)(0) - A_{v_-,x_-}(0)(0)| \tag{5.38}$$

$$+\left|\lim_{t\to+\infty}\left\{\int_0^t\left[g(\gamma(v_-)+\int_{-\infty}^{\tau}F(v_-s+x_-+y_-(s))ds)-g(\gamma(v_-)+\int_{-\infty}^{t}F(v_-s+x_-+y_-(s))ds)\right.\right.\right.$$
$$\left.\left.\left.-g(\gamma(v_-)+\int_{-\infty}^{\tau}F(v_-s+x_-)ds)+g(\gamma(v_-)+\int_{-\infty}^{t}F(v_-s+x_-)ds)\right]d\tau\right\}\right|.$$

Using (5.19), (5.20), (5.21), (5.24), (5.30), $\|y_-\|_T\le\rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)$, $T=+\infty$, and (5.38) we obtain (2.17c).

We shall prove (2.17b). First

$$v_-+k_{v_-,x_-}(y_-)=g(\gamma(v_-)+\int_{-\infty}^{+\infty}F(v_-s+x_-+y_-(s))ds). \tag{5.39}$$

Using the integral of motion E, we have $|v_-|=|v_-+k_{v_-,x_-}(y_-)|$ and applying γ to (5.39) we obtain

$$\frac{k_{v_-,x_-}(y_-)}{\sqrt{1-|v_-|^2/c^2}}=\int_{-\infty}^{+\infty}F(v_-s+x_-+y_-(s))ds. \tag{5.40}$$

From (5.40), (5.12), (5.13) and (4.11) and $\|y_-\|_T\le\rho(c,d,\beta_1,\alpha,|v_-|,|x_-|,r)$, $T=+\infty$, we obtain (2.17b)

Lemma 2.3 is proved.

6. Proofs of Theorems 3.2 and Proposition 1.1

Let $(\theta,x)\in T\mathbb{S}^{d-1}$, α, d, c, β_1, β_2 be fixed.

We shall use

$$\left|\frac{1}{s}\int_{-\infty}^{u}F(x+\tau\theta)d\tau\right|\le\frac{\beta_1\sqrt{d}}{\alpha(s/\sqrt{2})(1+|x|/\sqrt{2})^{\alpha}} \tag{6.1a}$$

for $s\in]0,c[$ and $u\in]-\infty,0]$; replacing θ by $-\theta$ in (6.1a), we obtain

$$\left|\frac{1}{s}\int_{u}^{+\infty}F(x+\tau\theta)d\tau\right|\le\frac{\beta_1\sqrt{d}}{\alpha(s/\sqrt{2})(1+|x|/\sqrt{2})^{\alpha}} \tag{6.1b}$$

for $s\in]0,c[$ and $u\in[0,+\infty[$.

We prove (6.1a). As $\theta x=0$, the following formula is valid:

$$|x+w\theta|\ge|x|/\sqrt{2}+|w|/\sqrt{2}, \tag{6.2}$$

for any $w\in\mathbb{R}$. Then estimate (6.1a) follows from (4.1), (6.2) and (4.6).

Before proving Theorem 3.2, we need introduce three Lemmas and prove them.

Lemma 6.1. *There exists integrable* $\tilde g_{c,d,\beta_0,\beta_1,\alpha,|x|} :]-\infty,0]\to[0,+\infty[$ *such that*

$$\left|(1+\delta_1(c,\theta,x,s,u))^{-\frac12}-1-\frac{V(x+u\theta)\sqrt{1-\frac{s^2}{c^2}}}{c^2}\right|\le \tilde g_{c,d,\beta_0,\beta_1,\alpha,|x|}(u)(1-s^2/c^2), \tag{6.3}$$

for $u\in]-\infty,0]$ *and* $s<c,\ s\ge z_2(c,d,\beta_1,\alpha,|x|)$, *and where*

$$\delta_1(c,\theta,x,s,u)=\frac{-2V(x+u\theta)\sqrt{1-\frac{s^2}{c^2}}+(\frac{1}{s^2}-\frac{1}{c^2})\left|\int_{-\infty}^u F(x+\tau\theta)d\tau\right|^2}{c^2}\ge-\frac34, \tag{6.4}$$

for $u\in]-\infty,0]$ *and* $s<c,\ s\ge z_2(c,d,\beta_1,\alpha,|x|)$.

Proof of Lemma 6.1.
Let $s\in]0,c[, s\ge z_2(c,d,\beta_1,\alpha,|x|)$ and $u\in]-\infty,0]$.
From (6.1a) and the definition of $z_2(c,d,\beta_1,\alpha,|x|)$ (see (1.7d)) it follows that

$$\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac1s\int_{-\infty}^u F(x+\tau\theta)d\tau\right|\ge\frac{s}{2\sqrt{1-s^2/c^2}}. \tag{6.5}$$

Expanding the square of the norm we obtain:
(6.6)

$$\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac1s\int_{-\infty}^u F(x+\tau\theta)d\tau\right|^2=\frac{s^2}{1-s^2/c^2}-\frac{2V(x+u\theta)}{\sqrt{1-s^2/c^2}}+\left|\frac1s\int_{-\infty}^u F(x+\tau\theta)d\tau\right|^2.$$

Using (6.5) and (6.6), we obtain

$$\begin{aligned}\delta_1(c,\theta,x,s,u)=&\frac{1+\frac{1}{c^2}\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac1s\int_{-\infty}^u F(x+\tau\theta)d\tau\right|^2}{1+\frac{s^2}{c^2-s^2}}-1\ge\frac{1+\frac{s^2}{4(c^2-s^2)}}{1+\frac{s^2}{c^2-s^2}}-1\\ \ge&-3/4.\end{aligned} \tag{6.7}$$

Moreover, from the definition of $\delta_1(c,\theta,x,s,u)$, (1.2), (6.1a) and the hypothesis $s\ge z_2(c,d,\beta_1,\alpha,|x|)$, $s<c$, it follows that

$$\begin{aligned}|\delta_1(c,\theta,x,s,u)|\le&\sqrt{1-s^2/c^2}\left[\frac{\beta_0 2(1+|x|/\sqrt2-u/\sqrt2)^{-\alpha}}{c^2}\right.\\ &\left.+\frac{\sqrt{1-z_2(c,d,\beta_1,\alpha,|x|)^2/c^2}\beta_1^2 d2(1+|x|/\sqrt2-u/\sqrt2)^{-2\alpha}}{z_2(c,d,\beta_1,\alpha,|x|)^2c^2\alpha^2}\right].\end{aligned} \tag{6.8}$$

Using Taylor expansion of the map $]-1,+\infty[\to\mathbb{R}$, $\delta\mapsto(1+\delta)^{-1/2}$ at $\delta=0$, we obtain that

$$(1+\delta_1(c,\theta,x,s,u))^{-\frac{1}{2}} - 1 - \frac{V(x+u\theta)\sqrt{1-\frac{s^2}{c^2}}}{c^2} = -\frac{1-s^2/c^2}{2s^2c^2}\left|\int_{-\infty}^{u} F(x+\tau\theta)d\tau\right|^2$$
$$+\frac{3}{4}\int_0^1 (1-w)(1+w\delta_1(c,\theta,x,s,u))^{-5/2}dw\ \delta_1(c,\theta,x,s,u)^2. \tag{6.9}$$

We estimate the first term of the right-hand side of (6.9) with the help of (6.1a).We estimate the second term of the right-hand side of (6.9) with the help of (6.7) and (6.8). Using also the inequality $s \geq z_2(c,d,\beta_1,\alpha,|x|)$, we finally obtain

$$\left|(1+\delta_1(c,\theta,x,s,u))^{-\frac{1}{2}} - 1 - \frac{V(x+u\theta)\sqrt{1-\frac{s^2}{c^2}}}{c^2}\right| \leq \tilde{g}_{c,d,\beta_0,\beta_1,\alpha,|x|}(u)$$

where

$$\begin{aligned}\tilde{g}_{c,d,\beta_0,\beta_1,\alpha,|x|}(u) =& \frac{d\beta_1^2}{c^2 z_2(c,d,\beta_1,\alpha,|x|)^2\alpha^2(1+|x|/\sqrt{2}-u/\sqrt{2})^{2\alpha}} \\ &+ 4^{5/2}\frac{3}{2c^4}\Big[\beta_0(1+|x|/\sqrt{2}-u/\sqrt{2})^{-\alpha} \\ &+ \frac{\sqrt{1-z_2(c,d,\beta_1,\alpha,|x|)^2/c^2}\beta_1^2 d(1+|x|/\sqrt{2}-u/\sqrt{2})^{-2\alpha}}{z_2(c,d,\beta_1,\alpha,|x|)^2\alpha^2}\Big]^2.\end{aligned}$$

Lemma 6.1 is proved.

Lemma 6.2. *Let $\beta > 0$, $s \in]0,c[$, $s \geq z_2(c,d,\beta_1,\alpha,|x|)$. Then there exists a positive real number $k_{\beta,c,d,\beta_1,\alpha,|x|}$ such that*

$$\left|\left(1+\frac{1-s^2/c^2}{s^2c^2}\left|\int_{-\infty}^{+\infty} F(x+\tau\theta)d\tau\right|^2\right)^{-\beta} - 1\right| \leq (1-s^2/c^2)k_{\beta,c,d,\beta_1,\alpha,|x|}.$$

Proof of Lemma 6.2. We define

$$\delta_2(c,\theta,x,s) = \frac{1-s^2/c^2}{s^2c^2}\left|\int_{-\infty}^{+\infty} F(x+\tau\theta)d\tau\right|^2 \geq 0. \tag{6.10}$$

Using (6.1) and $s \geq z_2(c,d,\beta_1,\alpha,|x|)$, we obtain

$$\delta_2(c,\theta,x,s) \leq (1-s^2/c^2)\frac{d\beta_1^2 8}{c^2 z_2(c,d,\beta_1,\alpha,|x|)^2\alpha^2(1+|x|/\sqrt{2})^{2\alpha}}. \tag{6.11}$$

Using the Taylor expansion of the map $]-1,+\infty[\to \mathbb{R}$, $\delta \mapsto (1+\delta)^{-\beta}$ at $\beta = 0$ and using (6.10), we obtain

$$(1+\delta_2(c,\theta,x,s))^{-\beta} - 1 = -\beta\delta_2(c,\theta,x,s)\int_0^1 (1+w\delta_2(c,\theta,x,s))^{-(\beta+1)}dw. \tag{6.12}$$

From (6.10), (6.11) and (6.12) it follows that

$$|(1+\delta_2(c,\theta,x,s))^{-\beta} - 1| \leq \beta\delta_2(c,\theta,x,s) \leq (1-s^2/c^2)k_{\beta,c,d,\beta_1,\alpha,|x|},$$

where

$$k_{\beta,c,d,\beta_1,\alpha,|x|} = \frac{\beta d\beta_1^2 8}{c^2 z_2(c,d,\beta_1,\alpha,|x|)^2\alpha^2(1+|x|/\sqrt{2})^{2\alpha}}.$$

Lemma 6.2 is proved.

We always suppose that $(\theta,x)\in T\mathbb{S}^{d-1}$, α, d, c, β_1, β_2 are fixed. Let $s\in]0,c[$, $s \geq z_2(c,d,\beta_1,\alpha,|x|)$, $u\in[0,+\infty[$. we define

$$A(c,\theta,x,s,u) = (1+t(c,\theta,x,s,u))^{-\frac{1}{2}}. \tag{6.13}$$

where

$$t(c,\theta,x,s,u) = \frac{1+\frac{1}{c^2}\left|\frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s}\int_{-\infty}^{u} F(x+\tau\theta)d\tau\right|^2}{1+\frac{1}{c^2}\left|\frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s}\int_{-\infty}^{+\infty} F(x+\tau\theta)d\tau\right|^2} - 1. \tag{6.14}$$

Expanding square of the norms in the numerator and denominator of the fraction of the right-hand side of (6.14), we obtain that

$$\begin{aligned} t(c,\theta,x,s,u) =& \frac{-2V(x+u\theta)\sqrt{1-s^2/c^2} + \frac{1-s^2/c^2}{s^2}\left|\int_u^{+\infty} F(x+\tau\theta)d\tau\right|^2}{\left(1+\frac{(1-s^2/c^2)}{s^2c^2}\left|\int_{-\infty}^{+\infty} F(x+\tau\theta)d\tau\right|^2\right)c^2} \\ & - \frac{\frac{2(1-s^2/c^2)}{s^2}\int_u^{+\infty} F(x+\tau\theta)d\tau . \int_{-\infty}^{+\infty} F(x+\tau\theta)d\tau}{\left(1+\frac{(1-s^2/c^2)}{s^2c^2}\left|\int_{-\infty}^{+\infty} F(x+\tau\theta)d\tau\right|^2\right)c^2}. \end{aligned} \tag{6.15}$$

Lemma 6.3. *There exists $h_{c,d,\beta_0,\beta_1,\alpha,|x|} : [0,+\infty[\to [0,+\infty[$ an integrable function such that for $s\in]0,c[$, $s\geq z_2(c,d,\beta_1,\alpha,|x|)$, $u\in[0,+\infty[$,*

$$\left|A(c,\theta,x,s,u) - 1 - V(x+u\theta)\frac{\sqrt{1-s^2/c^2}}{c^2}\right| \leq (1-s^2/c^2)h_{c,d,\beta_0,\beta_1,\alpha,|x|}(u).$$

Proof of Lemma 6.3. We first look for a lower bound for $t(c,\theta,x,s,u)$. The following estimate is valid

$$\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int_{-\infty}^{u}F(x+\tau\theta)d\tau\right| \geq \left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int_{-\infty}^{+\infty}F(x+\tau\theta)d\tau\right| - \left|\frac{1}{s}\int_{u}^{+\infty}F(x+\tau\theta)d\tau\right|. \tag{6.16}$$

From (6.1) it follows that

$$\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int_{-\infty}^{+\infty}F(x+\tau\theta)d\tau\right| \geq \frac{s}{\sqrt{1-s^2/c^2}} - \frac{2\beta_1\sqrt{d}}{\alpha(s/\sqrt{2})(1+|x|/\sqrt{2})^{\alpha}}. \tag{6.17}$$

Using first (6.1b) and then $s\geq z_2(c,d,\beta_1,\alpha,|x|)$ and (6.17) we obtain

$$\begin{aligned}\left|\frac{1}{s}\int_{u}^{+\infty}F(x+\tau\theta)d\tau\right| \leq & \frac{\beta_1\sqrt{d}}{(s/\sqrt{2})\alpha(1+|x|/\sqrt{2})^{\alpha}}\\ \leq & \frac{1}{6}\left(\frac{s}{\sqrt{1-s^2/c^2}} - \frac{2\beta_1\sqrt{d}}{\alpha(s/\sqrt{2})(1+|x|/\sqrt{2})^{\alpha}}\right)\\ \leq & \frac{1}{6}\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int_{-\infty}^{+\infty}F(x+\tau\theta)d\tau\right|.\end{aligned} \tag{6.18}$$

From (6.16) and (6.18) it follows that

$$\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int_{-\infty}^{u}F(x+\tau\theta)d\tau\right| \geq \frac{5}{6}\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int_{-\infty}^{+\infty}F(x+\tau\theta)d\tau\right|. \tag{6.19}$$

Using (6.14) and (6.19) we obtain

$$t(c,\theta,x,s,u) \geq \frac{25}{36}-1=-\frac{11}{36}. \tag{6.20}$$

Now we look for an upper bound for $t(c,\theta,x,s,u)$. The right-hand side of (6.15) consists of a substraction of two fractions whose denominator is greater than c^2 and this implies

$$\begin{aligned}|t(c,\theta,x,s,u)| \leq & c^{-2}\Bigg|-2V(x+u\theta)\sqrt{1-s^2/c^2}+\frac{1-s^2/c^2}{s^2}\left|\int_{u}^{+\infty}F(x+\tau\theta)d\tau\right|^2\\ & -\frac{2(1-s^2/c^2)}{s^2}\int_{u}^{+\infty}F(x+\tau\theta)d\tau.\int_{-\infty}^{+\infty}F(x+\tau\theta)d\tau\Bigg|.\end{aligned} \tag{6.21}$$

Thus, using (1.2), (6.1), (4.7) and the fact $s \geq z_2(c,d,\beta_1,\alpha,|x|)$, we obtain

$$\begin{aligned}|t(c,\theta,x,s,u)| \leq & c^{-2}\sqrt{1-s^2/c^2}\Big[2\beta_0(1+|x|/\sqrt{2}+u/\sqrt{2})^{-\alpha} \\ & + \frac{\sqrt{1-z_2(c,d,\beta_1,\alpha,|x|)^2/c^2}}{z_2(c,d,\beta_1,\alpha,|x|)^2}\frac{d\beta_1^2 2}{\alpha^2(1+|x|/\sqrt{2}+u/\sqrt{2})^{2\alpha}} \\ & + \frac{\sqrt{1-z_2(c,d,\beta_1,\alpha,|x|)^2/c^2}}{z_2(c,d,\beta_1,\alpha,|x|)^2}\frac{d\beta_1^2 8}{\alpha^2(1+|x|/\sqrt{2}+u/\sqrt{2})^{\alpha}(1+|x|/\sqrt{2})^{\alpha}}\Big].\end{aligned} \tag{6.22}$$

Using (6.13), (6.20), the Taylor expansion of the map $]-1,+\infty[\mapsto \mathbb{R}$, $\delta \mapsto (1+\delta)^{-1/2}$ at $\delta = 0$ and (6.15), we obtain

$$\begin{aligned}&\left|A(c,\theta,x,s,u) - 1 - \frac{V(x+u\theta)\sqrt{1-s^2/c^2}}{c^2}\right| \\ &= \left|\frac{1}{2}t(c,\theta,x,s,u) + \frac{3}{4}\int_0^1 (1-w)(1+wt(c,\theta,x,s,u))^{-\frac{5}{2}}dw\, t(c,\theta,x,s,u)^2 - \frac{V(x+u\theta)\sqrt{1-s^2/c^2}}{c^2}\right| \\ &\leq \left|\frac{V(x+u\theta)\sqrt{1-s^2/c^2}}{c^2}\left[1-\left(1+\frac{1-s^2/c^2}{c^2s^2}\left|\int_{-\infty}^{+\infty}F(x+\tau\theta)d\tau\right|^2\right)^{-1}\right]\right| \\ &+\frac{1}{2}\frac{\frac{1-s^2/c^2}{s^2}\left|\int_u^{+\infty}F(x+\tau\theta)d\tau\right|^2 + \frac{2(1-s^2/c^2)}{s^2}\int_u^{+\infty}|F(x+\tau\theta)|d\tau.\int_{-\infty}^{+\infty}|F(x+\tau\theta)|d\tau}{\left(1+\frac{(1-s^2/c^2)}{s^2c^2}\left|\int_{-\infty}^{+\infty}F(x+\tau\theta)d\tau\right|^2\right)c^2} \\ &+\frac{3}{8}\left(\frac{25}{36}\right)^{-\frac{5}{2}}t(c,\theta,x,s,u)^2.\end{aligned} \tag{6.23}$$

We use Lemma 6.2, conditions (1.2) and the fact that $s \geq z_2(c,d,\beta_1,\alpha,|x|)$ to estimate the first term of the right-hand side of the inequality (6.23). In order to estimate the second term of the right-hand side of the inequality (6.23), we use the fact that the denominator is greater than c^2, and we also use (6.1), (4.7), the fact that $s \geq z_2(c,d,\beta_1,\alpha,|x|)$. We estimate the third term of the right-hand side of the inequality with (6.22). Thus we obtain

$$\left|A(c,\theta,x,s,u) - 1 - \frac{V(x+u\theta)\sqrt{1-s^2/c^2}}{c^2}\right| \leq (1-s^2/c^2)h_{c,d,\beta_0,\beta_1,\alpha,|x|}(u),$$

where

$$\begin{aligned}&h_{c,d,\beta_0,\beta_1,\alpha,|x|}(u) = \\ &\frac{1}{c^2}(1+|x|/\sqrt{2}+u/\sqrt{2})^{-\alpha}\Big\{\beta_0 k_{1,c,d,\beta_1,\alpha,|x|}\sqrt{1-z_2(c,d,\beta_1,\alpha,|x|)^2/c^2}\end{aligned}$$

$$+\frac{\beta_1^2 d}{\alpha^2 z_2(c,d,\beta_1,\alpha,|x|)^2}\Big((1+|x|/\sqrt{2}+u/\sqrt{2})^{-\alpha}+4(1+|x|/\sqrt{2})^{-\alpha}\Big)$$
$$+\frac{3}{2c^2}(\frac{25}{36})^{-\frac{5}{2}}\Big[\beta_0(1+|x|/\sqrt{2}+u/\sqrt{2})^{-\alpha/2}$$
$$+\frac{d\beta_1^2\sqrt{1-z_2(c,d,\beta_1,\alpha,|x|)^2/c^2}}{z_2(c,d,\beta_1,\alpha,|x|)^2\alpha^2(1+|x|/\sqrt{2}+u/\sqrt{2})^{\alpha/2}}$$
$$\times\Big((1+|x|/\sqrt{2}+u/\sqrt{2})^{-\alpha}+4(1+|x|/\sqrt{2})^{-\alpha}\Big)\Big]^2\Big\}.$$

Lemma 6.3 is proved.

Proof of Theorem 3.2.

Let $(\theta,x)\in T\mathbb{S}^{d-1}$, $\alpha,\ d,\ c,\ \beta_1,\ \beta_2,\ s\in]0,c[,\ s\ge z_2(c,d,\beta_1,\alpha,|x|)$ be fixed. We shall study the asymptotics of $l_{s\theta,x}(0)$ which is defined by formula (2.13b).

First we look for the asymptotics of

$$\int\limits_{-\infty}^{0}\left[g(\gamma(s\theta)+\int\limits_{-\infty}^{\tau}F(us\theta+x)du)-s\theta\right]d\tau.$$

By changes of variables, we obtain

$$\int\limits_{-\infty}^{0}\left[g(\gamma(s\theta)+\int\limits_{-\infty}^{\tau}F(us\theta+x)du)-s\theta\right]d\tau=$$

$$\int\limits_{-\infty}^{0}\left[\frac{\frac{\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s^2}\int\limits_{-\infty}^{\tau}F(u\theta+x)du}{\sqrt{1+c^{-2}\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int\limits_{-\infty}^{\tau}F(u\theta+x)du\right|^2}}-\theta\right]d\tau. \tag{6.24}$$

Expanding the square of the norm in the denominator of the fraction under the integral in (6.24), the denominator becomes

$$\left(1+c^{-2}\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int\limits_{-\infty}^{\tau}F(u\theta+x)du\right|^2\right)^{-1/2}=(1+\delta_1(c,\theta,x,s,u))^{-\frac{1}{2}}\times(1-s^2/c^2)^{\frac{1}{2}}, \tag{6.25}$$

where δ_1 is defined by formula (6.4). We define

$$\Lambda_1(\theta,x,s)=\left|(1-s^2/c^2)^{-\frac{1}{2}}\int\limits_{-\infty}^{0}\left[g(\gamma(s\theta)+\int\limits_{-\infty}^{\tau}F(us\theta+x)du)-s\theta\right]d\tau\right.$$
$$\left.-c^{-2}\int_{-\infty}^{0}V(\tau\theta+x)d\tau\theta-s^{-2}\int_{-\infty}^{0}\int_{-\infty}^{\tau}F(u\theta+x)dud\tau\right|. \tag{6.26}$$

From (6.26), (6.24) and (6.25), it follows that

$$\Lambda_1(\theta,x,s) \le \int_{-\infty}^{0} \Big|(1+\delta_1(c,\theta,x,s,u))^{-\frac{1}{2}}$$

$$-1-c^{-2}V(\tau\theta+x)\sqrt{1-s^2/c^2}\theta\Big|$$

$$\times\left(\frac{1}{\sqrt{1-s^2/c^2}}+s^{-2}\int_{-\infty}^{\tau}|F(u\theta+x)|du\right)d\tau$$

$$+\int_{-\infty}^{0}\left|(1+c^{-2}V(\tau\theta+x)\sqrt{1-s^2/c^2})\left(\frac{\theta}{\sqrt{1-s^2/c^2}}+s^{-2}\int_{-\infty}^{\tau}F(u\theta+x)du\right)\right.$$

$$\left.-\frac{\theta}{\sqrt{1-s^2/c^2}}-c^{-2}V(\tau\theta+x)\theta-s^{-2}\int_{-\infty}^{\tau}F(u\theta+x)du\right|d\tau. \tag{6.27}$$

We estimate the first integral of the right-hand side of (6.27) by the use of Lemma 6.1. Therefore expanding the first product under the second integral of the form $\int_{-\infty}^{0}$ of the right-hand side of (6.27), we obtain

$$\Lambda_1(\theta,x,s) \le \sqrt{1-s^2/c^2}\int_{-\infty}^{0}\left[\tilde g_{c,d,\beta_0,\beta_1,\alpha,|x|}(\tau)\left(1+s^{-2}\sqrt{1-s^2/c^2}\int_{-\infty}^{\tau}|F(u\theta+x)|du\right)\right.$$

$$\left.+\left|\frac{V(\tau\theta+x)}{s^2c^2}\int_{-\infty}^{\tau}F(u\theta+x)du\right|\right]d\tau. \tag{6.28}$$

We define $|V|_\infty = \sup_{y\in\mathbb{R}^d}|V(y)|$. Using (6.28), (1.2), (6.1a), (4.8) and the fact that $s \ge z_2(c,d,\beta_1,\alpha,|x|)$, we obtain

$$\Lambda_1(\theta,x,s) \le \sqrt{1-s^2/c^2}\left[\left(1+\frac{\sqrt{1-z_2(c,d,\beta_1,\alpha,|x|)^2/c^2}}{z_2(c,d,\beta_1,\alpha,|x|)^2}\frac{\beta_1\sqrt{d}\sqrt{2}}{\alpha(1+|x|/\sqrt{2})^\alpha}\right)\int_{-\infty}^{0}\tilde g_{c,d,\beta_0,\beta_1,\alpha,|x|}(\tau)d\tau\right.$$

$$\left.+\frac{\beta_1\sqrt{d}2|V|_\infty}{z_2(c,d,\beta_1,\alpha,|x|)^2c^2\alpha(\alpha-1)(1+|x|/\sqrt{2})^{\alpha-1}}\right]. \tag{6.29}$$

Now we look for the asymptotics of

$$\int_0^{+\infty}\left[g(\gamma(s\theta)+\int_{-\infty}^{\tau}F(us\theta+x)du)-g(\gamma(s\theta)+\int_{-\infty}^{+\infty}F(us\theta+x)du)\right]d\tau.$$

By changes of variables, we obtain

$$\int_0^{+\infty}\left[g(\gamma(s\theta)+\int_{-\infty}^{\tau}F(us\theta+x)du)-g(\gamma(s\theta)+\int_{-\infty}^{+\infty}F(us\theta+x)du)\right]d\tau$$

$$= \int\limits_0^{+\infty} \left[\frac{\frac{\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s^2} \int\limits_{-\infty}^{\tau} F(u\theta + x)du}{\sqrt{1 + c^{-2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int\limits_{-\infty}^{\tau} F(u\theta + x)du \right|^2}} \right.$$

$$\left. - \frac{\frac{\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s^2} \int\limits_{-\infty}^{+\infty} F(u\theta + x)du}{\sqrt{1 + c^{-2} \left| \frac{s\theta}{\sqrt{1-s^2/c^2}} + \frac{1}{s} \int\limits_{-\infty}^{+\infty} F(u\theta + x)du \right|^2}} \right] d\tau. \tag{6.30}$$

First we study the denominator of the first fraction under the integral of (6.30). From (6.14) and (6.13) it follows that

$$\left(1 + c^{-2} \left| \frac{s\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s} \int\limits_{-\infty}^{\tau} F(u\theta + x)du \right|^2 \right)^{-\frac{1}{2}}$$

$$= \left(1 + c^{-2} \left| \frac{s\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s} \int\limits_{-\infty}^{+\infty} F(u\theta + x)du \right|^2 \right)^{-\frac{1}{2}} A(c, \theta, x, s, \tau). \tag{6.31}$$

We define

$$\Lambda_2(c, \theta, x, s) = \left| (1 - s^2/c^2)^{-\frac{1}{2}} \int\limits_0^{+\infty} \left[g(\gamma(s\theta) + \int\limits_{\tau}^{+\infty} F(us\theta + x)du) - g(\gamma(s\theta) + \int\limits_{-\infty}^{+\infty} F(us\theta + x)du) \right] d\tau \right.$$

$$\left. -c^{-2} \int_0^{+\infty} V(\tau\theta + x)d\tau\theta + s^{-2} \int_0^{+\infty} \int_{\tau}^{+\infty} F(u\theta + x)dud\tau \right|. \tag{6.32}$$

From (6.30), (6.31) and (6.32) it follows that

$$\Lambda_2(c, \theta, x, s) \leq \Lambda_{2,1}(c, \theta, x, s) + \Lambda_{2,2}(c, \theta, x, s), \tag{6.33}$$

where

$$\Lambda_{2,1}(c, \theta, x, s) = \int_0^{+\infty} \left| (1 - s^2/c^2)^{-\frac{1}{2}} \left(1 + \frac{1}{c^2} \left| \frac{s\theta}{\sqrt{1 - s^2/c^2}} + \frac{1}{s} \int\limits_{-\infty}^{+\infty} F(u\theta + x)du \right|^2 \right)^{-\frac{1}{2}} \right.$$

$$\times\left(A(c,\theta,x,s,\tau)-1-V(\tau\theta+x)\frac{\sqrt{1-\frac{s^2}{c^2}}}{c^2}\right)\left(\frac{\theta}{\sqrt{1-\frac{s^2}{c^2}}}+\frac{1}{s^2}\int\limits_{-\infty}^{\tau}F(u\theta+x)du\right)\Bigg|d\tau, \tag{6.34a}$$

$$\Lambda_{2,2}(c,\theta,x,s)=\int_0^{+\infty}\Bigg|(1-s^2/c^2)^{-\frac{1}{2}}\left(1+\frac{1}{c^2}\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int\limits_{-\infty}^{+\infty}F(u\theta+x)du\right|^2\right)^{-\frac{1}{2}}$$

$$\times\Bigg[\left(1+V(\tau\theta+x)\frac{\sqrt{1-s^2/c^2}}{c^2}\right)\left(\frac{\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s^2}\int\limits_{-\infty}^{\tau}F(u\theta+x)du\right)$$

$$-\left(\frac{\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s^2}\int\limits_{-\infty}^{+\infty}F(u\theta+x)du\right)\Bigg]-\frac{V(\tau\theta+x)}{c^2}\theta+\frac{1}{s^2}\int\limits_{\tau}^{+\infty}F(u\theta+x)du\Bigg|d\tau. \tag{6.34b}$$

Let us estimate $\Lambda_{2,1}(c,\theta,x,s)$.

From Lemma 6.3 and (6.34a) it follows that

$$\Lambda_{2,1}(c,\theta,x,s)\leq\sqrt{1-s^2/c^2}\left(1+\frac{1}{c^2}\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int\limits_{-\infty}^{+\infty}F(u\theta+x)du\right|^2\right)^{-\frac{1}{2}}$$

$$\times\left(\frac{1}{\sqrt{1-s^2/c^2}}+\frac{1}{s^2}\int\limits_{-\infty}^{+\infty}|F(u\theta+x)|du\right)\times\int_0^{+\infty}h_{c,d,\beta_0,\beta_1,\alpha,|x|}(\tau)d\tau. \tag{6.35}$$

In addition, expanding the square of the norm, we obtain

$$\left(1+\frac{1}{c^2}\left|\frac{s\theta}{\sqrt{1-s^2/c^2}}+\frac{1}{s}\int\limits_{-\infty}^{+\infty}F(u\theta+x)du\right|^2\right)^{-\frac{1}{2}}=\sqrt{1-s^2/c^2}$$

$$\times\left(1+\frac{1-s^2/c^2}{s^2c^2}\left|\int\limits_{-\infty}^{+\infty}F(u\theta+x)du\right|^2\right)^{-\frac{1}{2}} \tag{6.36a}$$

$$\leq\sqrt{1-s^2/c^2}. \tag{6.36b}$$

Using (6.35), (6.36), (4.1), (6.2), (4.7) and the fact that $s\geq z_2(c,d,\beta_1,\alpha,|x|)$, we obtain

$$\Lambda_{2,1}(c,\theta,x,s)\leq\sqrt{1-s^2/c^2}\left(1+\frac{\beta_1\sqrt{d}\sqrt{1-z_2(c,d,\beta_1,\alpha,|x|)^2/c^2}2\sqrt{2}}{z_2(c,d,\beta_1,\alpha,|x|)^2\alpha(1+|x|/\sqrt{2})^\alpha}\right)$$

$$\times\int_0^{+\infty}h_{c,d,\beta_0,\beta_1,\alpha,|x|}(\tau)d\tau. \tag{6.37}$$

Let us estimate $\Lambda_{2,2}(c,\theta,x,s)$.

From (6.34b) and (6.36a) it follows that

$$\Lambda_{2,2}(c,\theta,x,s) \le \int_0^{+\infty} \left| \left(1 + \frac{1-s^2/c^2}{s^2c^2} \left| \int\limits_{-\infty}^{+\infty} F(u\theta+x)du \right|^2 \right)^{-\frac{1}{2}} - 1 \right|$$
$$\times \left| -\frac{1}{s^2} \int\limits_{\tau}^{+\infty} F(u\theta+x)du + \frac{V(\tau\theta+x)}{c^2}\theta + \frac{\sqrt{1-s^2/c^2}}{s^2c^2} V(\tau\theta+x) \int\limits_{-\infty}^{\tau} F(u\theta+x)du \right| d\tau$$
$$+ \int_0^{+\infty} \left| \frac{\sqrt{1-s^2/c^2}}{s^2c^2} V(\tau\theta+x) \int\limits_{-\infty}^{\tau} F(u\theta+x)du \right| d\tau. \tag{6.38}$$

Thus using Lemma 6.2, conditions (1.2), (4.1), (6.2), (4.6), (4.7) and the fact that $s \ge z_2(c,d,\beta_1,\alpha,|x|)$, it follows that

$$\Lambda_{2,2}(c,\theta,x,s) \le \sqrt{1-s^2/c^2} \left[\frac{k_{1/2,c,d,\beta_1,\alpha,|x|}\sqrt{1-z_2(c,d,\beta_1,\alpha,|x|)^2/c^2}}{(\alpha-1)(1+|x|/\sqrt{2})^{\alpha-1}} \left(\frac{\sqrt{d}\beta_1 2}{z_2(c,d,\beta_1,\alpha,|x|)^2\alpha} \right.\right.$$
$$\left. + \frac{\beta_0\sqrt{2}}{c^2} + \frac{\sqrt{d}\beta_0\beta_1 4\sqrt{1-z_2(c,d,\beta_1,\alpha,|x|)^2/c^2}}{z_2(c,d,\beta_1,\alpha,|x|)^2c^2\alpha(1+|x|/\sqrt{2})^{\alpha}} \right)$$
$$\left. + \frac{\sqrt{d}\beta_0\beta_1 4}{z_2(c,d,\beta_1,\alpha,|x|)^2c^2\alpha(\alpha-1)(1+|x|/\sqrt{2})^{2\alpha-1}} \right]. \tag{6.39}$$

From (2.13b), (6.26), (6.29), (6.32), (6.34), (6.35) and (6.39) it follows that there exists $C_{c,d,\beta_0,\beta_1,\alpha,|x|}$ such that

$$\left| \frac{l_{s\theta,x}(0)}{\sqrt{1-s^2/c^2}} - \frac{1}{c^2}PV(\theta,x)\theta + \frac{1}{s^2}\int_0^{+\infty}\int_{\tau}^{+\infty} F(x+u\theta)dud\tau - \frac{1}{s^2}\int_{-\infty}^{0}\int_{-\infty}^{\tau} F(x+u\theta)dud\tau \right|$$
$$\le \Lambda_1(c,\theta,x,s) + \Lambda_2(c,\theta,x,s)$$
$$\le C_{c,d,\beta_0,\beta_1,\alpha,|x|}\sqrt{1-s^2/c^2}. \tag{6.40}$$

The estimate (3.9) follows from (6.40).

Theorem 3.2 is proved.

Proof of Proposition 1.1. The item 1 follows immediately from

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t\theta+x) = \nabla V(t\theta+x)\theta, \text{ for all } (\theta,x) \in T\mathbb{S}^{d-1},\ t \in \mathbb{R}.$$

Proof of the item 2. Take

$$V(x) = \frac{x_1}{(1+|x|^2)^{\beta}}, \text{ for } x = (x_1,\ldots,x_d) \in \mathbb{R}^d, \beta > 1.$$

and take $(\theta, x) \in T\mathbb{S}^{d-1}$. By a straightforward calculation and using $\theta x = 0$, we obtain

$$\left(\int_{-\infty}^{0}\int_{-\infty}^{\tau} F(s\theta + x)dsd\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty} F(s\theta + x)dsd\tau + PV(\theta, x)\theta\right) \star x$$
$$= -4\beta\theta_1|x|^2 \int_0^{+\infty}\int_{\tau}^{+\infty} \frac{s}{(1 + s^2 + |x|^2)^{\beta+1}} dsd\tau$$
$$\neq 0 \text{ if and only if } x \neq 0 \text{ and } \theta_1 \neq 0,$$

where $\star$ denotes the scalar product.

Proof of the item 3. Let V be a spherical symmetric potential (i.e. V takes the form $m(|x|)$) that satisfies the conditions (1.2) (e.g. $V(x) = (1 + |x|^2)^{-\beta}$ where $\beta > \frac{1}{2}$). Then $m \in C^1(]0, +\infty[, \mathbb{R})$ and $\nabla V(x) = m'(|x|)\frac{x}{|x|}$. Let $(\theta, x) \in T\mathbb{S}^{d-1}$ and let $\theta^{\perp}$ be an orthogonal vector to θ. A straighforward calculation gives

$$F(s\theta + x)\theta^{\perp} = m'(\sqrt{s^2 + |x|^2})\frac{x \star \theta^{\perp}}{\sqrt{s^2 + |x|^2}}$$

for any $s \in \mathbb{R}$. Hence

$$\left(\int_{-\infty}^{0}\int_{-\infty}^{\tau} F(s\theta + x)dsd\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty} F(s\theta + x)dsd\tau + PV(\theta, x)\theta\right) \star \theta^{\perp} = 0. \tag{6.41}$$

The item 3 follows from the item 1 and formula (6.41).

Proposition 1.1 is proved.

Acknowledgement. This work was fulfilled in the framework of Ph. D. thesis researchs under the direction of R.G. Novikov.

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